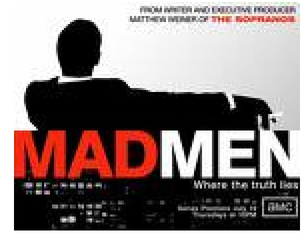
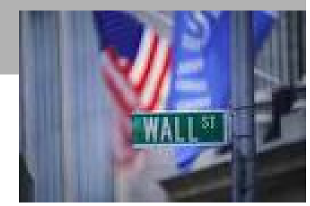
L3: Outline: CoAd Lectures

- **Introduction**
- L1 **Online advertising background**
- **Business models, Campaigns**
- **Technology and Economics** (L2)
  - Forward Markets
    - Gradient Descent, Operations research, LP, QP
  - Auction Theory and Game Theory
  - Spot Markets (L3)
    - ML, Ad quality, Ranking, Budgeting
- L4 **New Directions**
- **Challenges in online advertising**
- **Summary**

Business, Gold rush



Tech



Hot Areas

CoAd Lectures		
Friday	9/11/2009	10:30-12:00
Saturday	9/12/2009	8:30-10:00
Sunday	9/13/2009	8:30-10:00
Monday	9/14/2009	8:30-10:00

Course philosophy

- **Socratic Method (more inspiration than information)**
  - participation strongly encouraged (please state your name and affiliation)
- **Highly interactive and adaptable**
  - Questions welcome!!
- **Lectures emphasize intuition, less rigor and detail**
  - Build on lectures from other faculty
  - Background reading will provide more rigor & detail
- **Action Items**
  - Read suggested books first (and then papers), read/**write** Wikipedia, watch/**make** YouTube videos, take courses, participate in competitions, do internships, network
  - Prototype, simulate , publish, participate
  - Classic (core) versus trendy (applications)

Lecture 2: Homework

- **Email solutions to James.Shanahan__AT__gmail.com**
- **Exercises**
  – Find a local minimum of the function $f(x)=6x^5-8x^2+6$
  – Implement gradient descent version of Perceptron
  – Implement gradient descent version of OLS; show evolution of weight vector during training

Forward Markets

- **Linear Programming**
- **Quadratic Programming**
- **Allocation of Ads to Publisher real estate**
  - Give ads play in network
    - Optimize *revenue* subject to ….
- **Inventory Management**
  - Contract as many impressions as possible but don't oversell
- **Media Buyer (Arbitrage)**
  - Frame as a non-linear programming (NLP) problem
  - Talks to publisher
  - Determine publisher mix for network
    - Optimize *publisher mix* subject to constraints

Nonlinear Programming

- Nonlinear programming (NLP) is the process of solving a system of equalities and inequalities, collectively termed constraints, over a set of unknown real variables, along with an objective function to be maximized or minimized,
- where some of the constraints or the objective function are nonlinear.

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Lagrange Theorem [Lagrange 1788]

Objective function : z = f(x) **Equality Constraint**

subject to: $c_i(x) = 0; \; i \in [1, \ldots, m]$ **Goto Appendix A**

$$L(W, \lambda) = f(x) - \sum_i \lambda_i [c_i(x)]$$

1st Order Conds

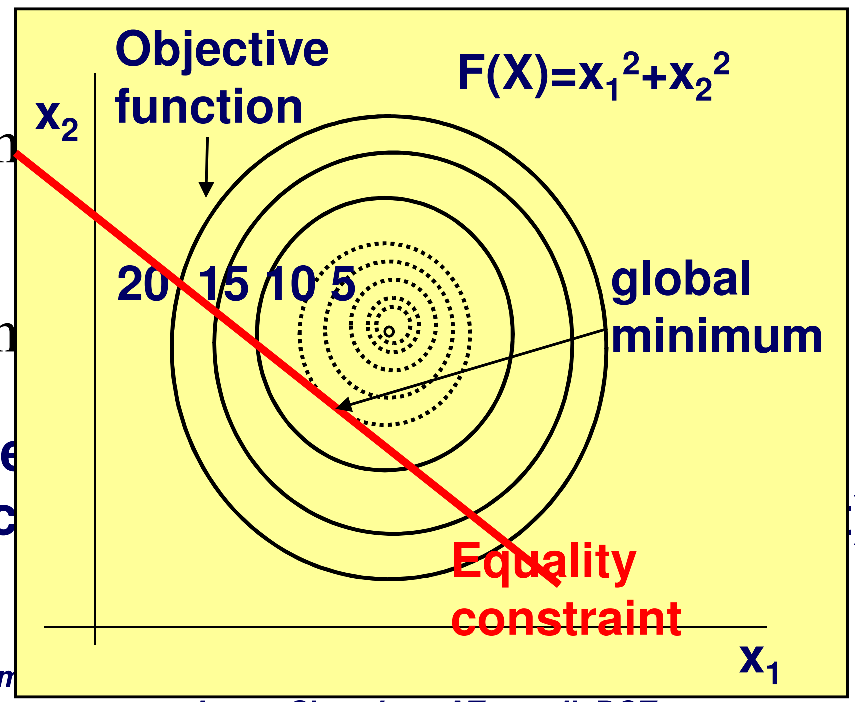
$$\frac{\partial P(\mathbf{w},\lambda)}{\partial \mathbf{w}_i} = 0 \text{ and } \frac{\partial P(\mathbf{w},\lambda)}{\partial \lambda_i} = 0$$

Optimum = W^*, λ^*;
Optimum if d'L= 0

2nd Order Conds (Min)

$$\frac{\partial^2 P(\mathbf{w}^*,\lambda^*)}{\partial \mathbf{w}_i} > 0 \text{ if } (\mathbf{w}^*,\lambda^*) \text{ is m}$$

$$\frac{\partial^2 P(\mathbf{w}^*,\lambda^*)}{\partial \mathbf{w}_i} < 0 \text{ if } (\mathbf{w}^*,\lambda^*) \text{ is m}$$



Results in a system of "n + m" (n variable
Solve these equations to determine critic).
Take second derivative and determine
if critical points are max or min

Quadratic Programming: Dual Soft SVM

Soft Margin SVM (Primal)

$$\min(W,b) = 0.5 \times \|w\|^2 + C\sum_{i=1}^{L}\xi_i$$

$$\text{subject to: } y_i(\langle W, X_i\rangle + b) + \xi_i \geq 1 \quad \forall\, i = 1,\ldots,L$$

$$\text{and } \xi_i \geq 0 \quad \forall\, i = 1,\ldots,L$$

Soft Margin SVM (Dual)

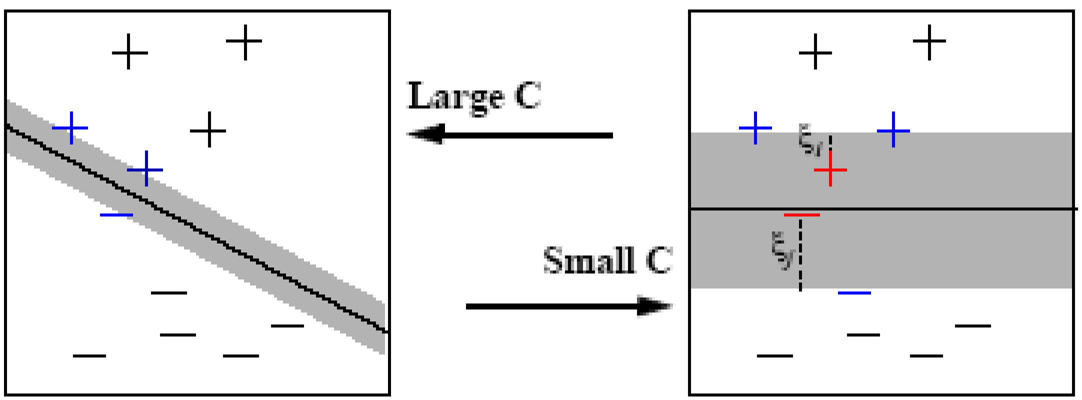
$$\max_{\alpha} W(\alpha) \sum_{i=1}^{L}\alpha_i - \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L} y_i y_j \langle X_i, X_j\rangle \alpha_i \alpha_j,$$

$$0 \leq \alpha_i \leq C, \quad \forall i = 1,\ldots,L$$

$$\sum_{i=1}^{L} y_i a_i = 0.$$

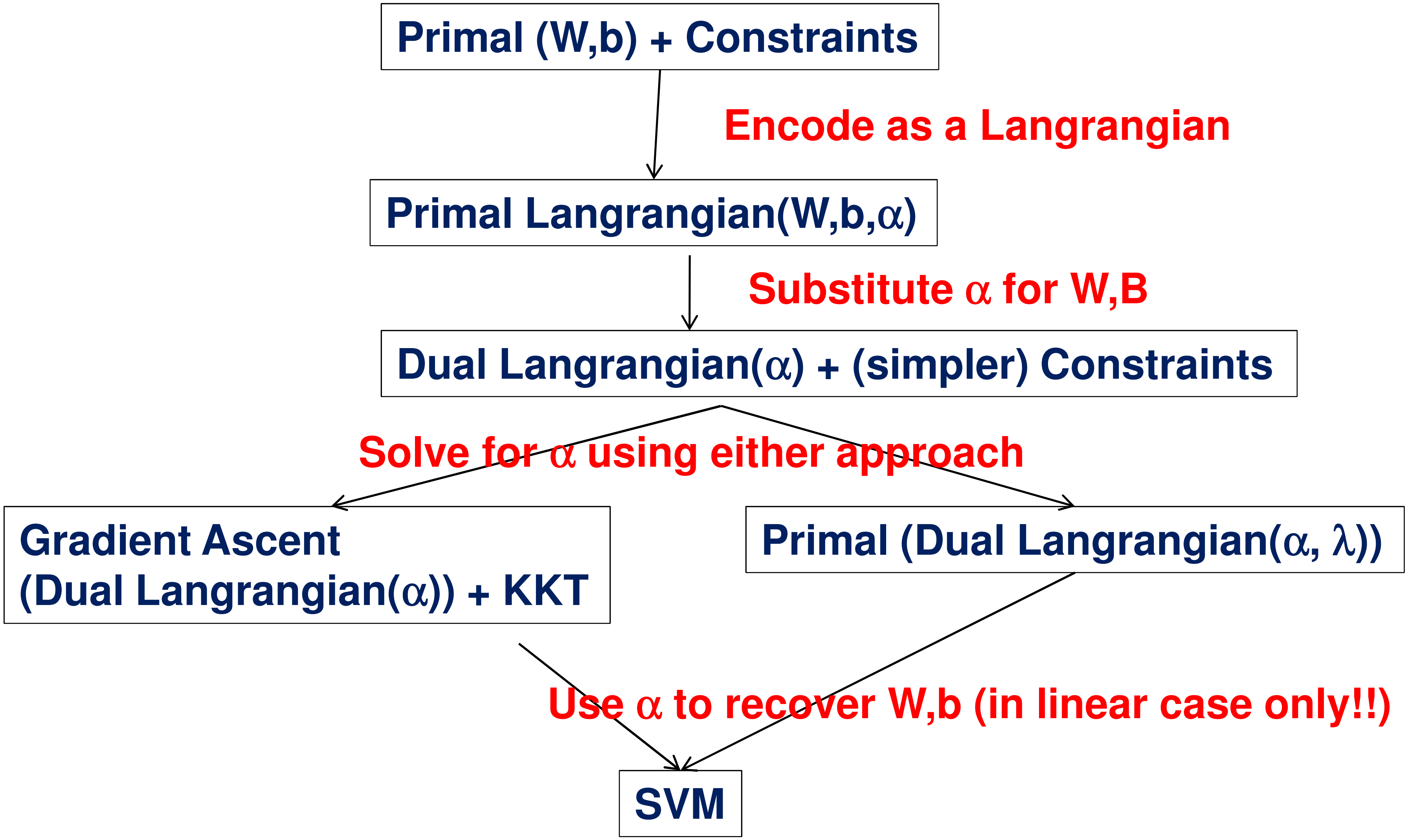
Upper bounds for α

- **Large C => Hard Margin (allow very few errors)**
- **Small C => allow a lot of slack and therefore large margin**



[Source: http://www.cs.cornell.edu/Courses/CS678/2003sp/slides/perceptron_4up.pdf]

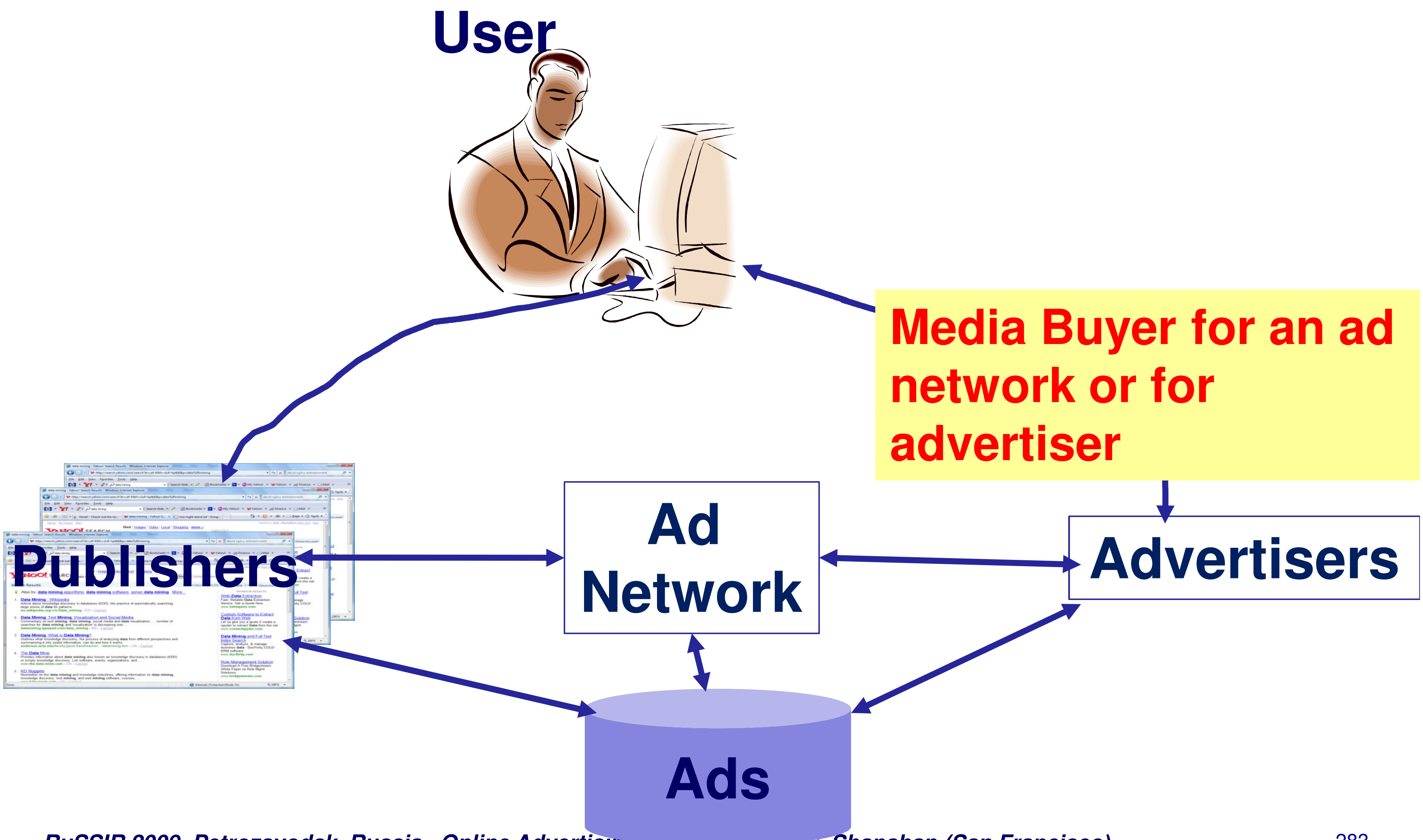
SVM Learning Algorithms



Forward Markets

- **Linear Programming**
- **Quadratic Programming**
- **Allocation of Ads to Publisher real estate**
  - Give ads play in network
    - Optimize *revenue* subject to ….
- **Inventory Management**
  - Contract as many impressions as possible but don't oversell
- **Media Buyer (Arbitrage)**
  - Buy media, buy keywords
  - Frame as a non-linear programming (NLP) problem
  - Talks to publisher (or search engine)
  - Determine publisher mix for network (or keyword mix)
    - Optimize *publisher mix* subject to constraints

Online Advertising: Media Buyer



Portfolio Optimization: Markowitz model

- **In the following slides, we will show how to model portfolio optimization as an NLP**
- **The key concept is that risk can be modeled using non-linear equations**
- **In, e.g., finance, one tradesoff risk and return. For a given rate of return, one wants to minimize risk.**
  - For a given rate of risk, one wants to maximize return.
  - Return is modeled as expected value.
  - Risk is modeled as variance (or standard deviation.)

Nobel Prize for Portfolio Mgt.[1990]

Press Release - The Sveriges Riksbank (Bank of Sweden) Prize in Economic Sciences in Memory of Alfred Nobel

KUNGL. VETENSKAPSAKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

16 October 1990

THIS YEAR'S LAUREATES ARE PIONEERS IN THE THEORY OF FINANCIAL ECONOMICS AND CORPORATE FINANCE

The Royal Swedish Academy of Sciences has decided to award the 1990 Alfred Nobel Memorial Prize in Economic Sciences with one third each, to

Professor **Harry Markowitz**, City University of New York, USA,
Professor **Merton Miller**, University of Chicago, USA,
Professor **William Sharpe**, Stanford University, USA,

for their pioneering work in the theory of financial economics.

Harry Markowitz is awarded the Prize for having developed the theory of portfolio choice;
William Sharpe, for his contributions to the theory of price formation for financial assets, the so-called, *Capital Asset Pricing Model* (CAPM); and
Merton Miller, for his fundamental contributions to the theory of corporate finance.

Summary
Financial markets serve a key purpose in a modern market economy by allocating productive resources among various areas of production. It is to a large extent through financial markets that saving in different sectors of the economy is transferred to firms for investments in buildings and machines. Financial markets also reflect firms' expected prospects and risks, which implies that risks can be spread and that savers and investors can acquire valuable information for their investment decisions.

The first pioneering contribution in the field of financial economics was made in the 1950s by Harry Markowitz who developed a theory for households' and firms' allocation of financial assets under uncertainty, the so-called theory of portfolio choice. This theory analyzes how wealth can be optimally invested in assets which differ in regard to their expected return and risk, and thereby also how risks can be reduced.

Portfolio allocation under uncertainty return-risk tradeoff [~1950]

[Slides adapted from http://www.princeton.edu/~rvdb/542/lectures/lec17.pdf]

7 The Markowitz Model–The Objective

Decision Variables: the fractions x_j.

Objective: maximize return, minimize risk.

Fundamental Lesson: can't simultaneously optimize two objectives.

Compromise: maximize a combination of reward and risk:

$$\text{reward}(x) - \mu\text{risk}(x)$$

Parameter μ is called risk aversion parameter.

$0 \leq \mu < \infty$

Large value for μ puts emphasis on risk minimization.

Small value for μ puts emphasis on reward maximization.

11 Interior-Point Methods for Quadratic Programming

Start with an optimization problem—in this case QP:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Use slack variables to make all inequality constraints into nonnegativities:

$$\begin{array}{ll} \text{minimize} & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax - w = b \\ & x, w \geq 0 \end{array}$$

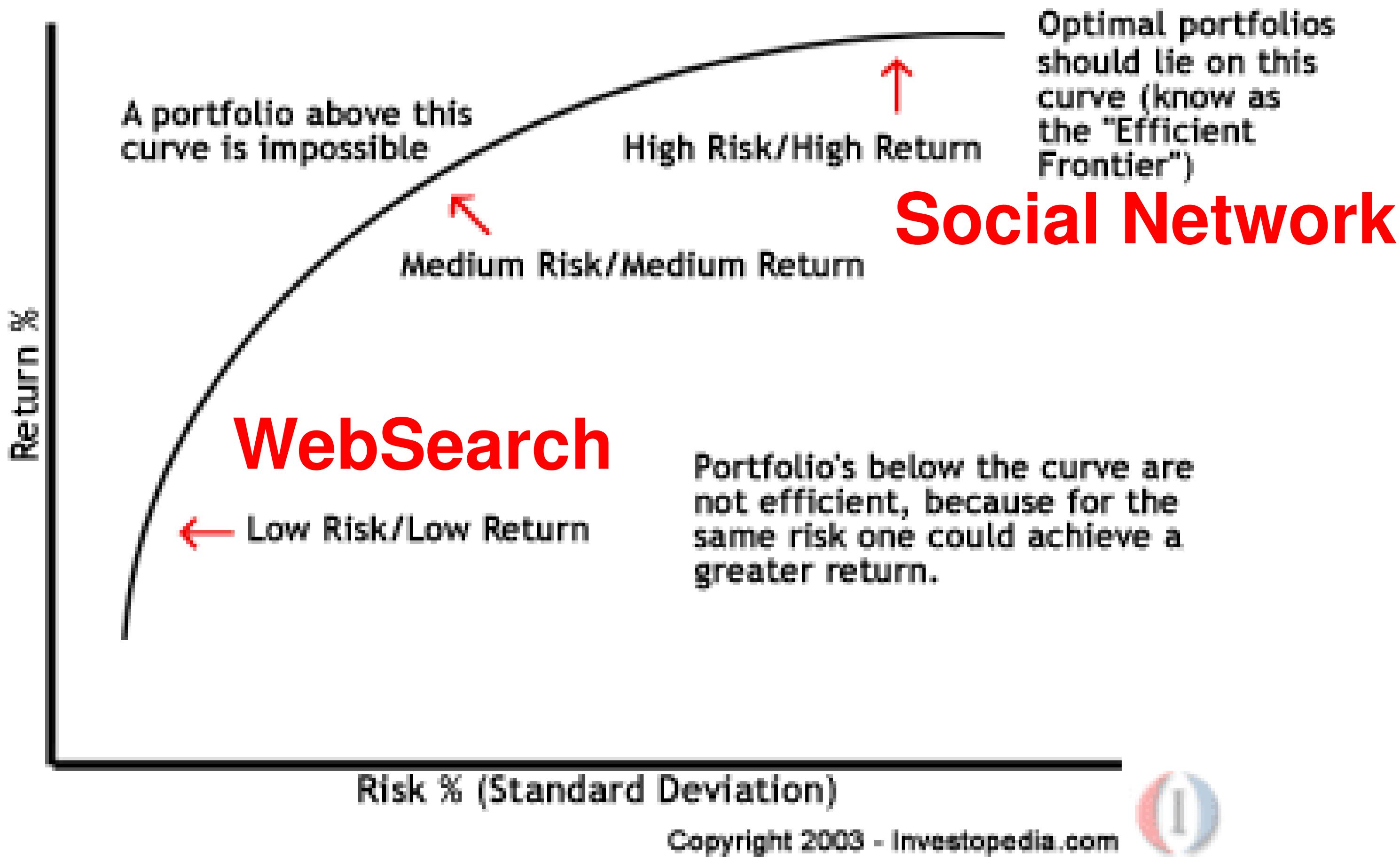
Replace nonnegativity constraints with logarithmic barrier terms in the objective:

Efficient Frontier

Varying μ produces the so-called efficient frontier.

Portfolios on the efficient frontier are reasonable.

Portfolios not on the efficient frontier can be strictly improved.


Optimal portfolios should lie on this curve (know as the "Efficient Frontier")
A portfolio above this curve is impossible
High Risk/High Return
Social Network
Medium Risk/Medium Return
Return %
WebSearch
Portfolio's below the curve are not efficient, because for the same risk one could achieve a greater return.
Low Risk/Low Return
Risk % (Standard Deviation)
Copyright 2003 - Investopedia.com

Variance-Covariance Matrix

	ROI on Publisher	Week#	MySpace	Forbes	Glam
1	ROI on Publisher	Week#	MySpace	Forbes	Glam
2		1	30.00%	22.50%	14.90%
3		2	10.30%	29.00%	26.00%
4		3	21.60%	21.60%	41.90%
5		4	-4.60%	-27.20%	-7.80%
6		5	-7.10%	14.40%	16.90%
7		6	5.60%	10.70%	-3.50%
8		7	3.80%	32.10%	13.30%
9		8	8.90%	30.50%	73.20%
10		9	9.00%	19.50%	2.10%
11		10	8.30%	39.00%	13.10%
12		11	3.50%	-7.20%	0.60%
13		12	17.60%	71.50%	90.80%
14					
15		Average	8.91%	21.37%	23.46%
16					
17					
18	Covariance		MySpace	Forbes	Glam
19	Matrix	MySpace	**0.0099**	0.0114	0.0120
20		Forbes	0.0114	**0.0535**	0.0508
21		Glam	0.0120	0.0508	**0.0864**

$$\overline{R}_i = \frac{1}{n}\sum_{t=1}^{n} R_i^t$$

$$= \frac{1}{n}\sum_{t=1}^{n}\left(R_i^t - \overline{R}_i\right)\left(R_j^t - \overline{R}_j\right)$$

How much of each publisher?

15		Average	8.91%	21.37%	23.46%				
16									
17									
18	Covariance		MySpace	Forbes	Glam				
19	Matrix	MySpace	**0.0099**	0.0114	0.0120				
20		Forbes	0.0114	**0.0535**	0.0508				
21		Glam	0.0120	0.0508	**0.0864**				
27									
28			MySpace	Forbes	Glam				
29	Decisions		53.01%	35.64%	11.35%				
30	Constraints	Min Return	8.91%	21.37%	23.46%	15.00%	≥	15%	My Goal
31		Portfolio	1	1	1	100.00%	=	100%	
32		MySpace	1			53.01%	≤	75%	
33		Forbes		1		35.64%	≤	75%	
34		Glam			1	11.35%	≤	75%	
35									
36	Minimize	Portfolio Variance =	X^TYX	0.0205					
37									

$$\bar{R}_i = \frac{1}{n}\sum_{t=1}^{n} R_i^t$$

$$= \frac{1}{n}\sum_{t=1}^{n} \left(R_i^t - \bar{R}_i\right)\left(R_j^t - \bar{R}_j\right)$$

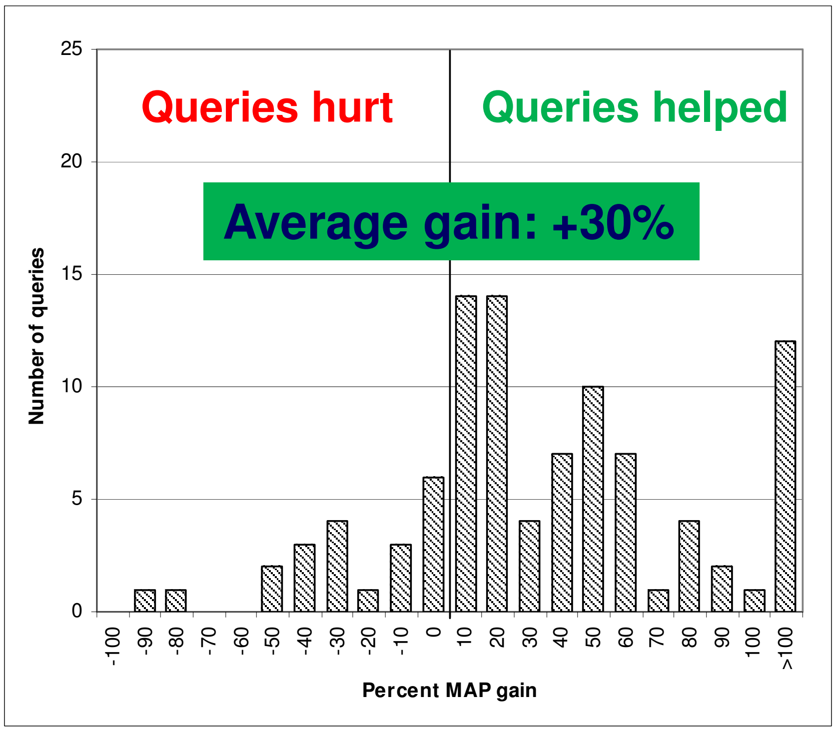
Quadratic Prog | PublisherSelection (2) | PublisherSelection

Select destination and press ENTER or choose Paste

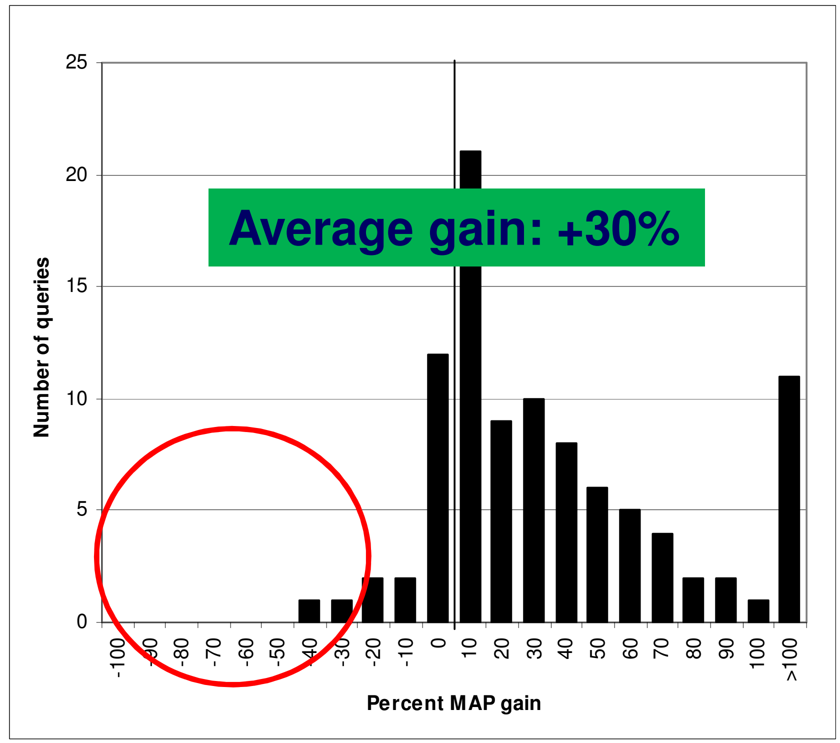
Applying Portfolio Mgt. to information retrieval

Want a robust query algorithm that almost never hurts, while preserving large average gains

Apply Markowitz to query expansion!



Query expansion: Current state-of-the-art method



Robust version using Markowitz

[Collins-Thompson, NIPS 2008]

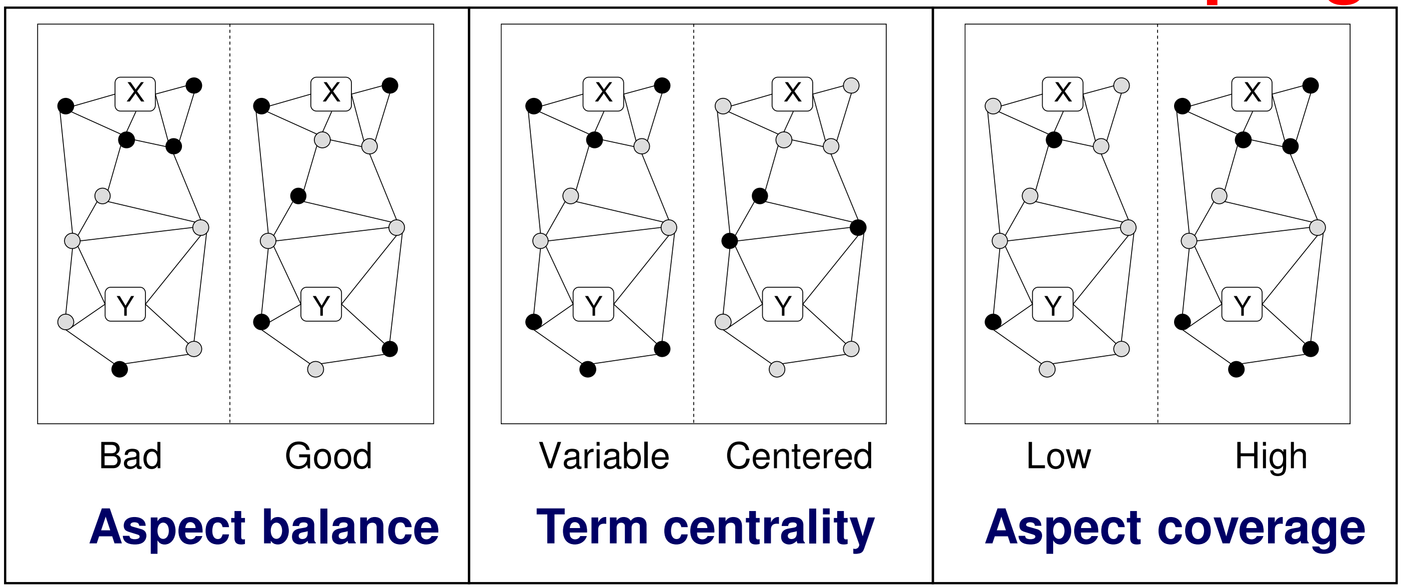
What is a good objective function for query expansion?

- **Markowitz: portfolio allocation under uncertainty**
- **[Collins-Thompson: NIPS 2008, PhD dissertation]**
- **Reward:**
  - Baseline provides initial weight vector *c*
  - Prefer words with higher c_i values: $R(x) = c^T x$
- **Risk:**
  - Model uncertainty in *c* using a covariance matrix Σ
  - Model uncertainty in Σ using regularized $\Sigma_\gamma = \Sigma + \gamma D$
  - Diagonal: captures individual term variance (centrality)
  - Off-diagonal: term covariance (co-occurrence)
- **Combined objective:**

$$U(x) = -R(x) + \kappa V(x) = -c^T x + \kappa x^T (\Sigma + \gamma D) x$$

[Collins-Thompson, NIPS 2008]

These conditions are complementary and can be combined with the objective into quadratic program



QMOD algorithm

$$\text{minimize} \quad -c^T x + \frac{\kappa}{2} x^T \Sigma_\gamma x \qquad \textit{Term relevance, centrality, risk}$$

$$\text{subject to} \quad Ax \leq \mu + \varsigma_\mu \qquad \textit{Aspect balance}$$

$$g_i^T x \geq \varsigma_i, \quad w_i \in Q \qquad \textit{Aspect coverage}$$

$$l_i \leq x_i \leq u_i, \; w_i \in Q \qquad \textit{Query term support}$$

$$0 \leq x \leq 1$$

Example solution output

Query: `parkinson's disease`

Baseline expansion

parkinson	0.996
disease	0.848
syndrome	0.495
disorders	0.492
parkinsons	0.491
patient	0.483
brain	0.360
patients	0.313
treatment	0.289
diseases	0.153
alzheimers	0.114
...*and 90 more*...	

Convex QMOD expansion

parkinson	0.9900
disease	0.9900
syndrome	0.2077
parkinsons	0.1350
patients	0.0918
brain	0.0256

(All other terms removed)

Scheduling Ads: Other Reported Work

- **Global Allocation Solutions (forward markets)**
  - Scheduling house ads: Decision Trees + Linear/I/Q Programming for ad selection with websites, e.g., AMEX [Poindexter.com]
  - Heuristics, Genetic Algorithms, Integer Programming, See Ali Amiri, Syam Menon: Efficient scheduling of Internet banner advertisements. ACM Trans. Internet Techn. 3(4): 334-346 (2003)

Forward Markets and Optimisation

In Summary

- **Gradient descent, LP, QP are fundamental**
  - Not only in advertising but also in ML, IR
- **Allocation of Ads to Publisher real estate**
  - Give ads play in network
    - Optimize *revenue* subject to ….
- **Inventory Management**
  - Contract as many impressions as possible but don't oversell
- **Media Buyer (Arbitrage)**
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Gradient Descent/LP/QP Reading Material

- **Duda, Hart, & Stork (2000). Pattern Classification, Wiley.**
- **Statistical machine learning, Friedman et al. 2001, Springer.**
- **Linear and Nonlinear Programming by David G. Luenberger, Yinyu Ye**
- **Linear Programming by Vašek Chvátal**
  - readable online (at least the first 3 chapters)
  - http://books.google.com/books?id=DN20_tW_BV0C&pg=PP1&dq=Linear+Programming,+by+Vasek+Chv%C3%A1tal&ei=4VegSZSQN53wkQSoyPWhCA#PPA41,M1
- **Introduction to Operations Research, 8/edition by Frederick S Hillier, Stanford University, Gerald J Lieberman, Stanford University, ISBN: 0073017795, Copyright year: 2005**
- **Chapters 2 and 3 of Schaum's Outline of Operations Research, (second edition) by Richard Bronson, Govindasami Naadimuthu**
- **Atsuyoshi Nakamura and Naoki Abe (http://www.research.ibm.com/people/n/nabe/JECR05-NA.pdf) Improvements to the Linear Programming based Scheduling of Web Advertisements, *Journal of Electronic Commerce Research, 5(1), 75-98, 2005.***
- **•M. Langheinrich, A. Nakamura, N. Abe, T. Kamba and Y. Koseki, (http://www8.org/w8-papers/2b-customizing/unintrusive/unintrusive.html) Unintrusive customization techniques for Web advertising, *Computer Networks* 31,pp.1259-1272, 1999. Targeted Internet Advertising Using Predictive Clustering and Linear Programming**

Forward Markets Bibliography

- Linear and Nonlinear Programming by David G. Luenberger, Yinyu Ye
- Introduction to Operations Research, 8/edition by Frederick S Hillier, Stanford University, Gerald J Lieberman, Stanford University, ISBN: 0073017795, Copyright year: 2005
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- •M. Langheinrich, A. Nakamura, N. Abe, T. Kamba and Y. Koseki, (http://www8.org/w8-papers/2b-customizing/unintrusive/unintrusive.html) Unintrusive customization techniques for Web advertising, *Computer Networks* 31,pp.1259-1272, 1999. Targeted Internet Advertising Using Predictive Clustering and Linear Programming
- David Maxwell Chickering, David Heckerman, Christopher Meek, John C. Platt, and Bo Thiesson, Targeted Internet Advertising Using Predictive Clustering and Linear Programming, 2004

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- **Business models**
- **Creating an online ad campaign**
- **Technology and Economics**
  - Advertisers (optimizing ROI thru ads and ad placement)
  - Publishers (optimizing revenue and consumer satisfaction)
    - Forward/Future Markets
    - Spot Markets
      - Background
      - Auction Systems, Game Theory
      - Ad Quality
      - Budgeting
- **New Directions**
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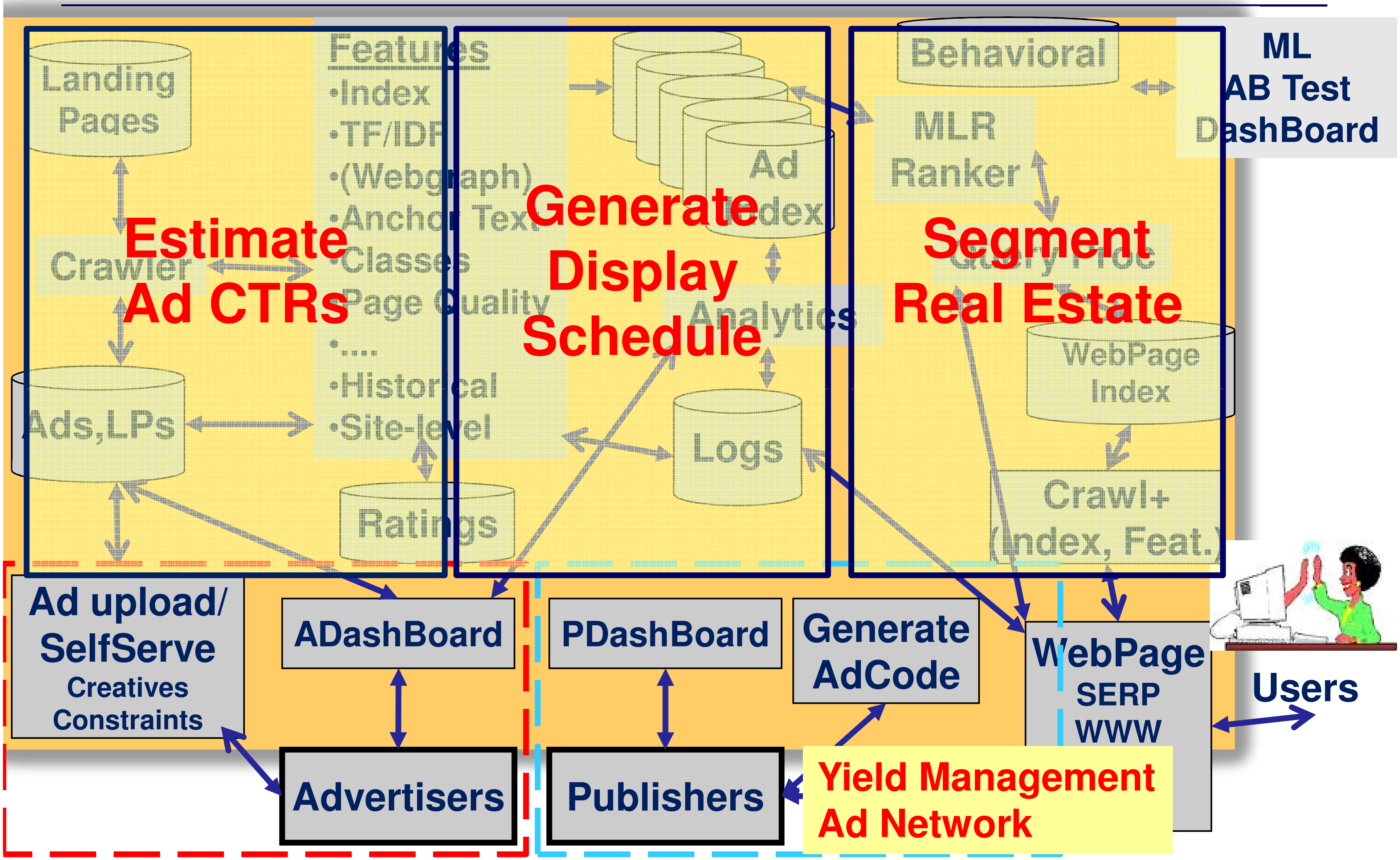
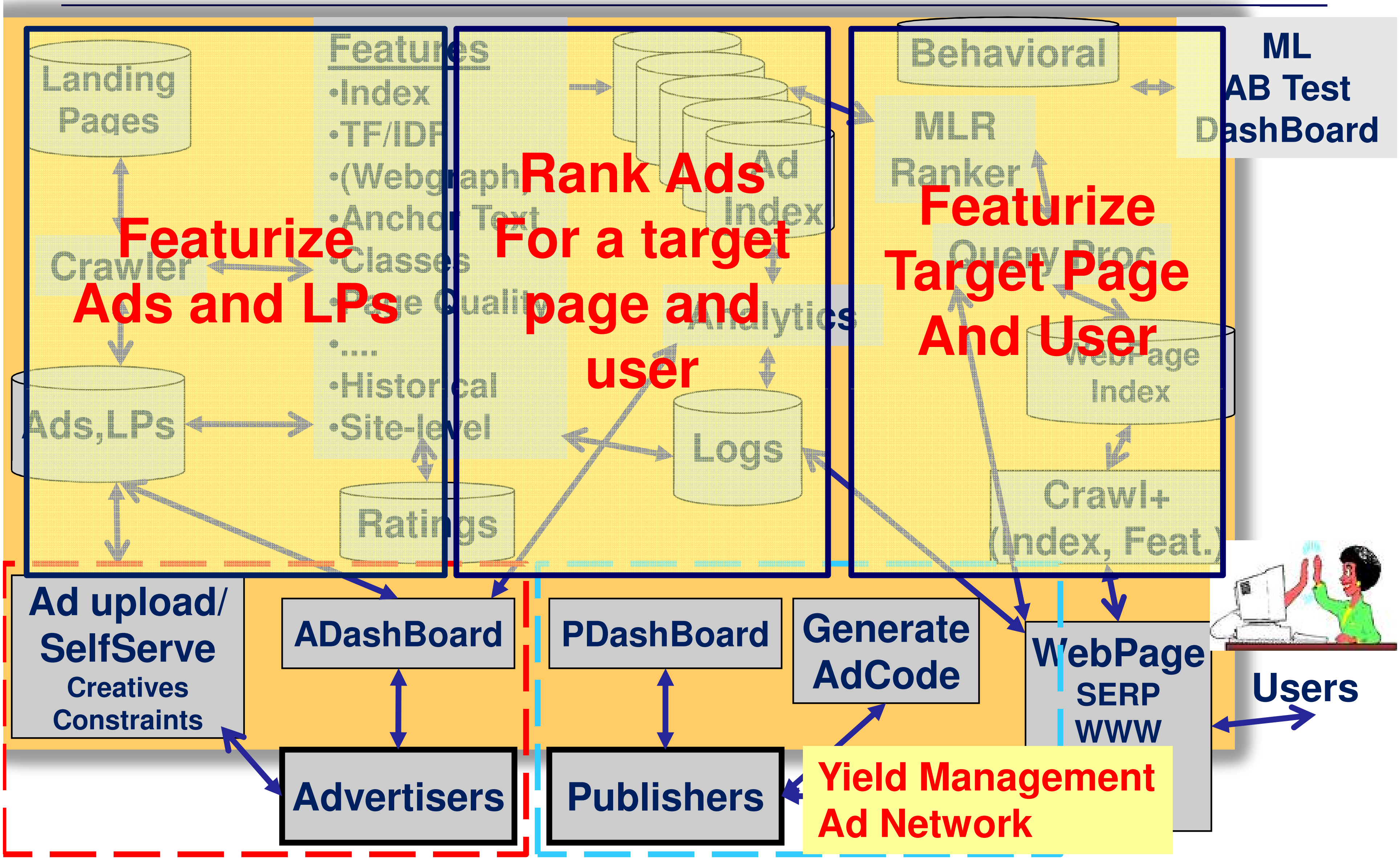
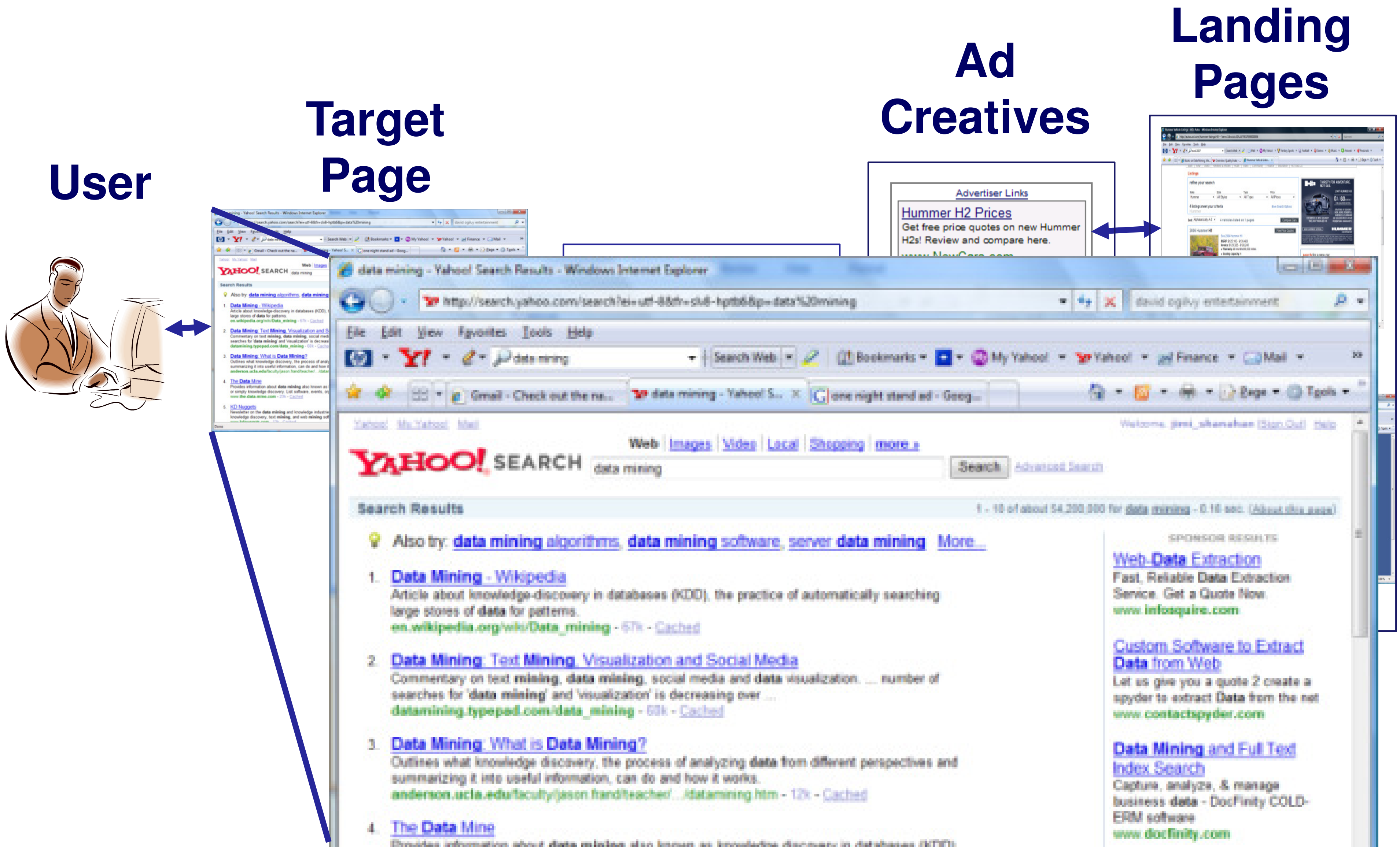
Ad Network Architecture: Forward Market



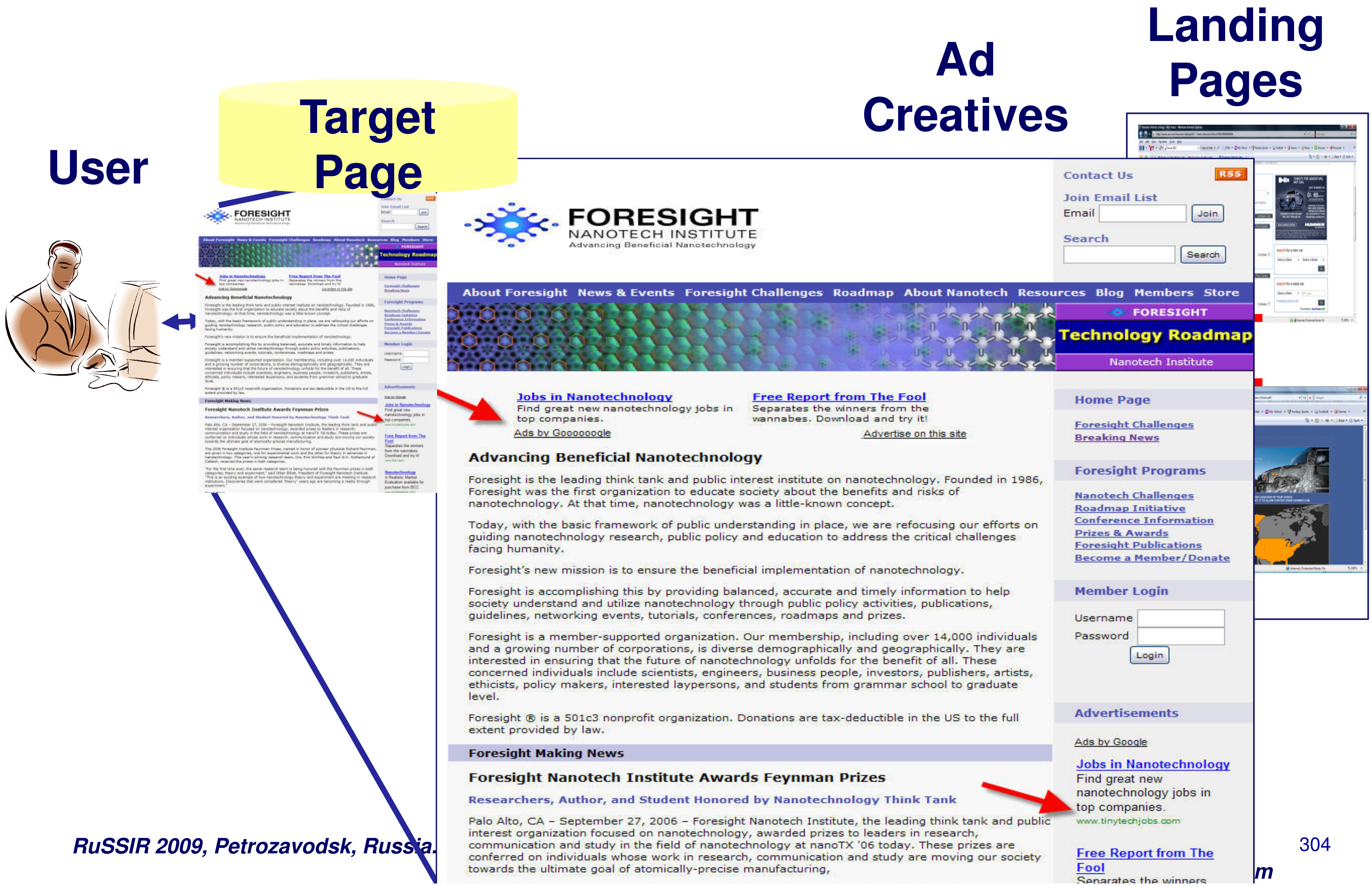
Ad Network Architecture: Spot Market



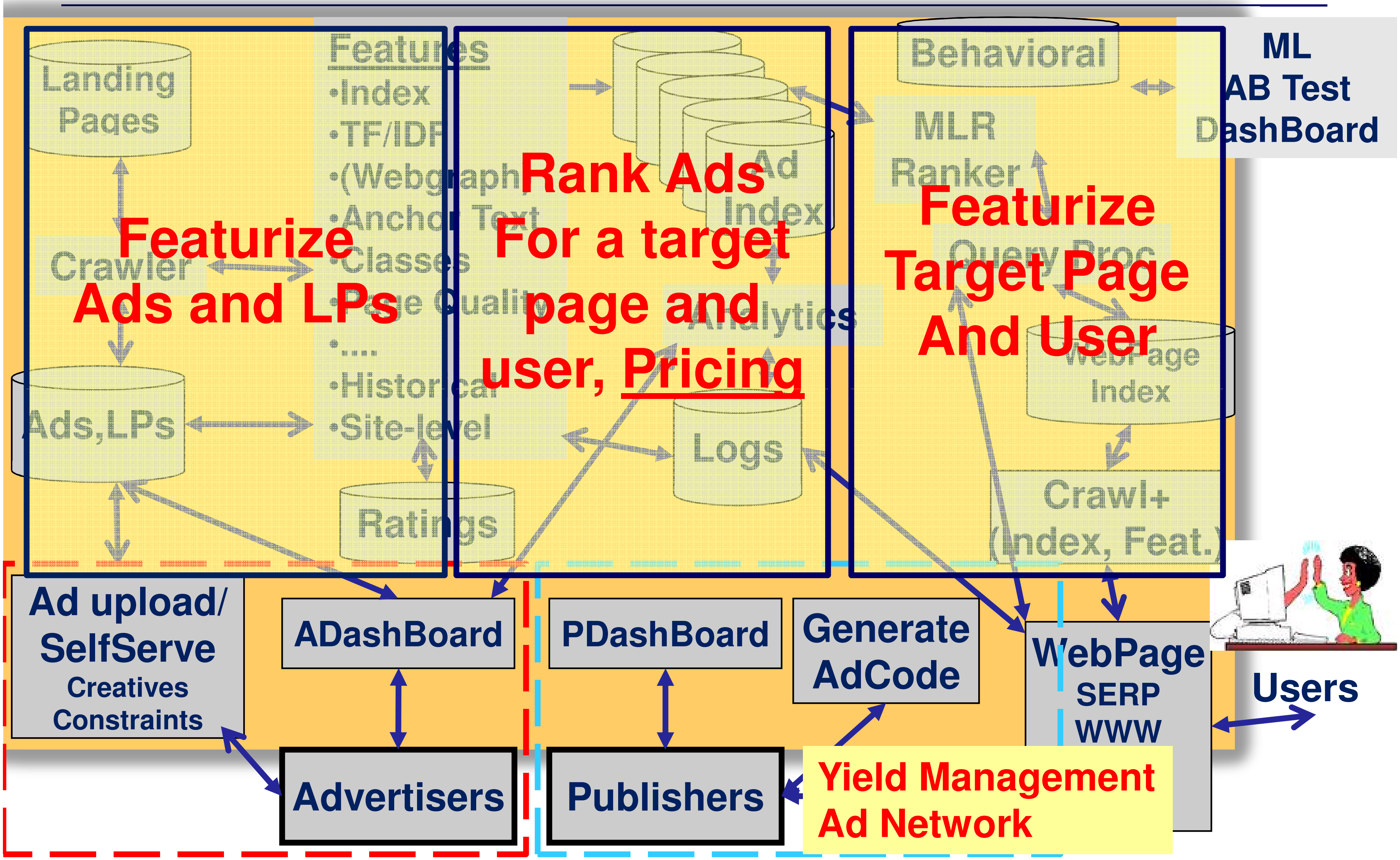
CPC Paid Search (KW Market place)



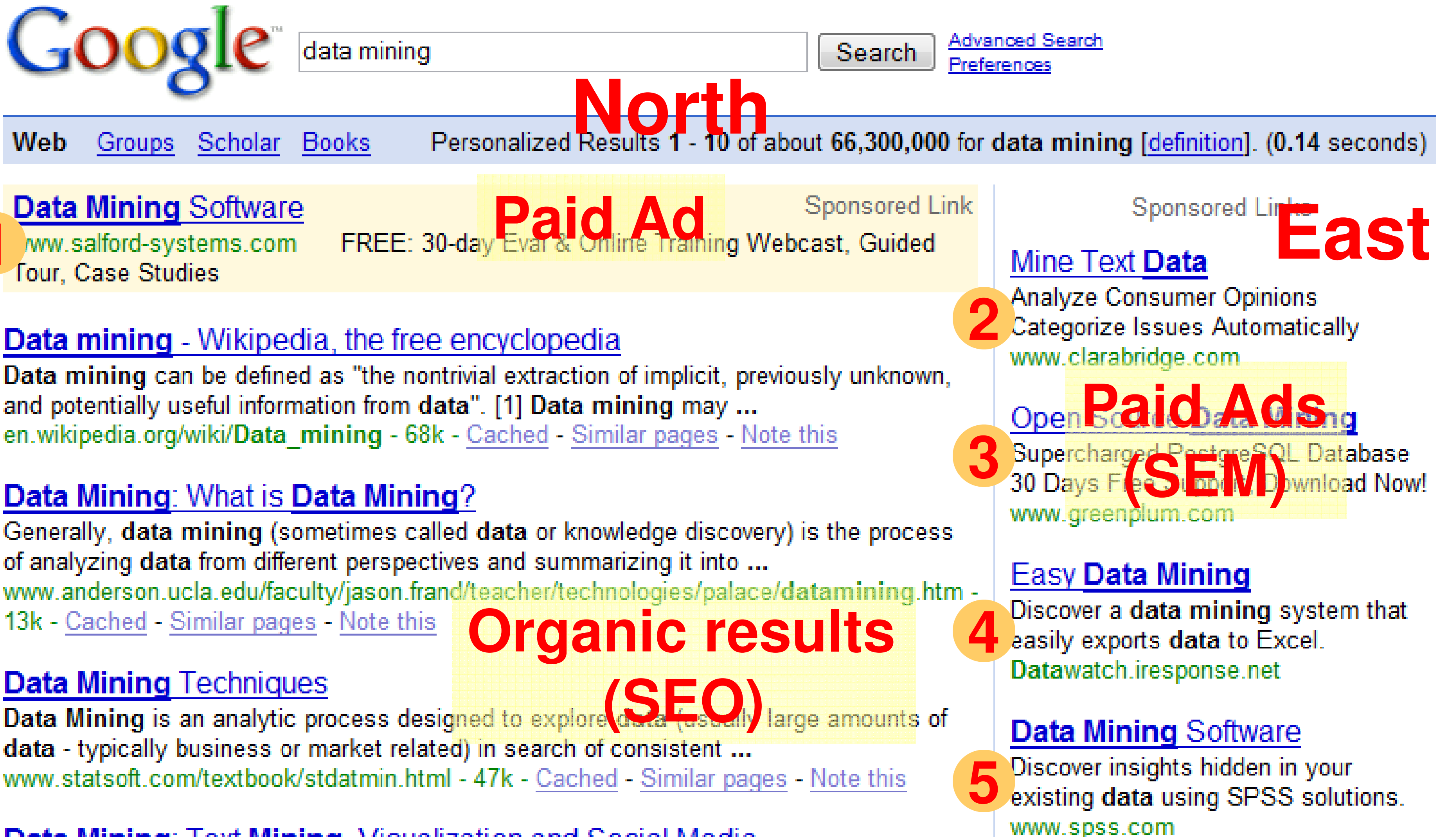
CPC Contextual vs. CPC Paid Search



Ad Network Architecture: Spot Market



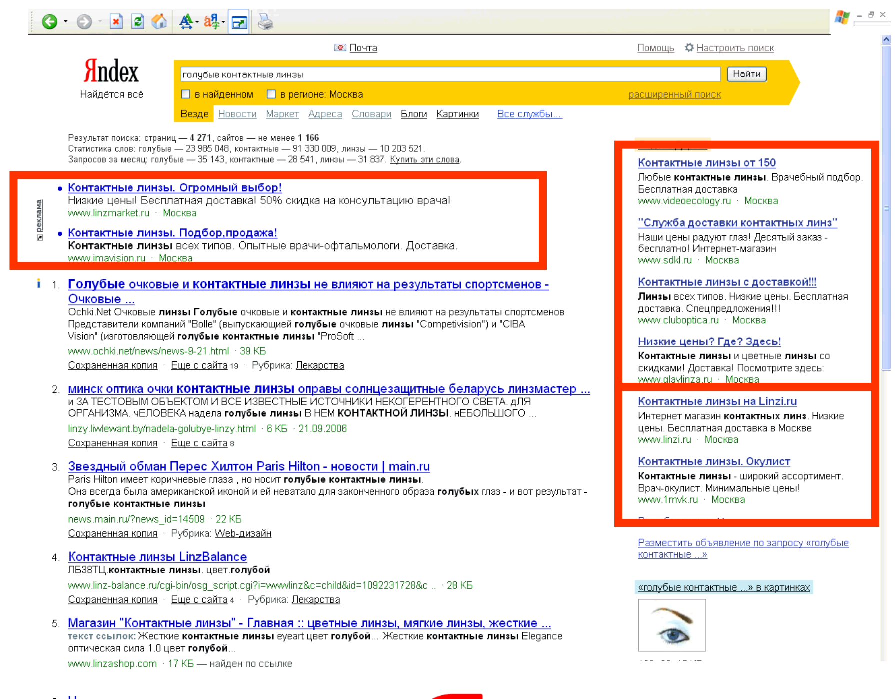
Sponsored Search (vs Contextual)



Ads are clearly distinguishable from the actual search results and they rotate

Yandex Direct on Yandex SERP

- **premium position**(3 **advertising maximum)**
  - Static show
- **Guaranteed placement position on the right** (4 **Top position on the right)**
  - Static show
- **Dynamic Show**(5 **advertising max)**



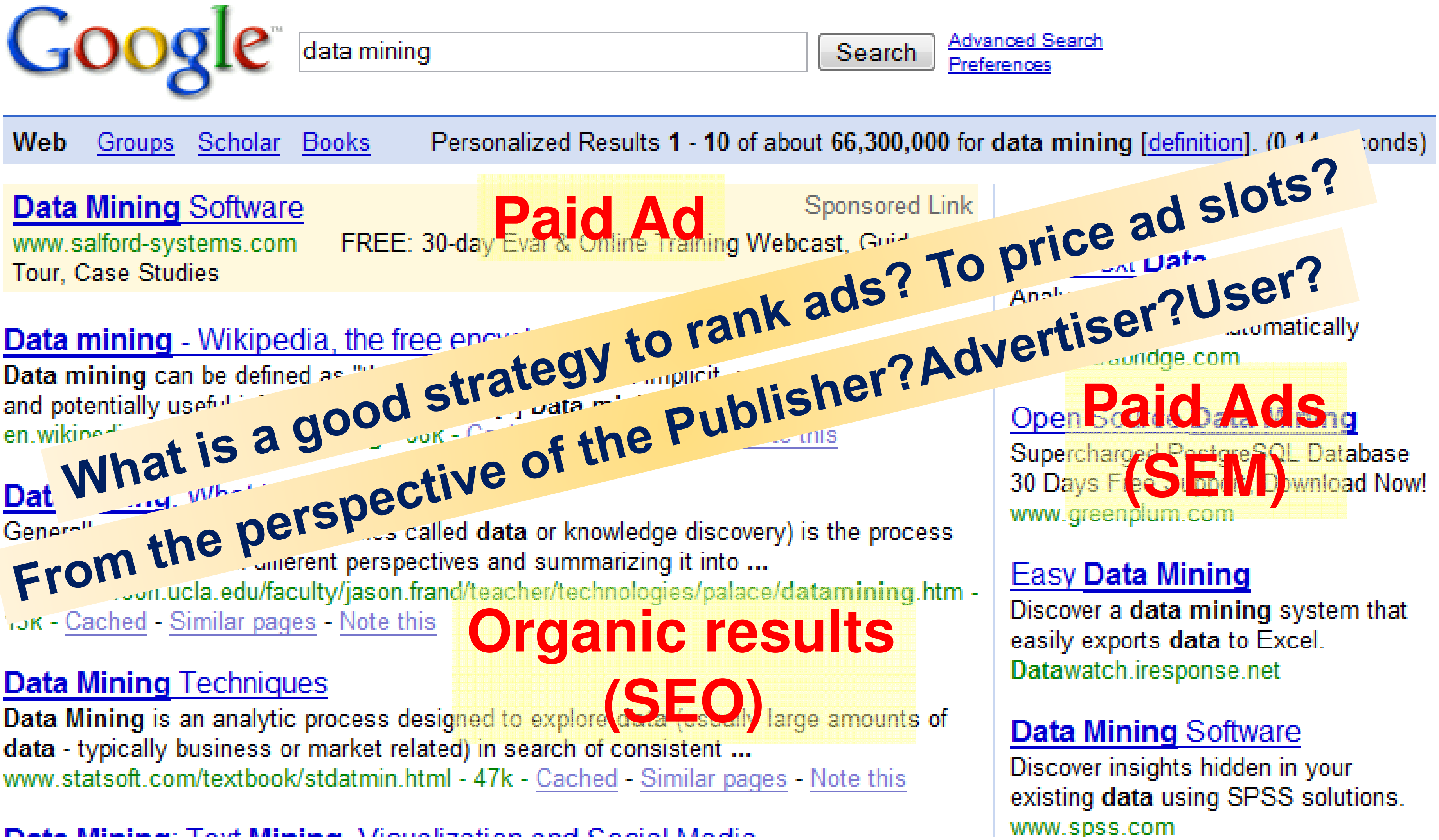
Яндекс

[Evgeny Lomize, Bogdan Garkushin, direct.Yandex.com]

Organic Search Results Boosting

- **Search engine optimization**
  - Independent third parties help tune a client's website potentially yielding a higher rankings on the organic rankings
- **Paid inclusion (e.g., Yahoo)**
  - Lets Web site owners submit information about their pages to search engines. They're guaranteed inclusion in the search engine's index, but aren't given any assurances regarding how their pages will be ranked.
- **Paid placement programs guaranteed top listings**
  - Addressed search engine spam (on organic results) in the early days of web search
  - Goto.com [1997]
  - Advertisers bid on exact search terms; vetted by editors

Sponsored Search (vs Contextual)



Ads are clearly distinguishable from the actual search results and they rotate

Ad Ranking and Pricing

- Search engines and more generally ad networks need a system for allocating the positions/slots to ads

- Preset price and randomly rotate
- Keyword bid price?
- IR-based approach?
- Click through rates?
- Combinations of the above?

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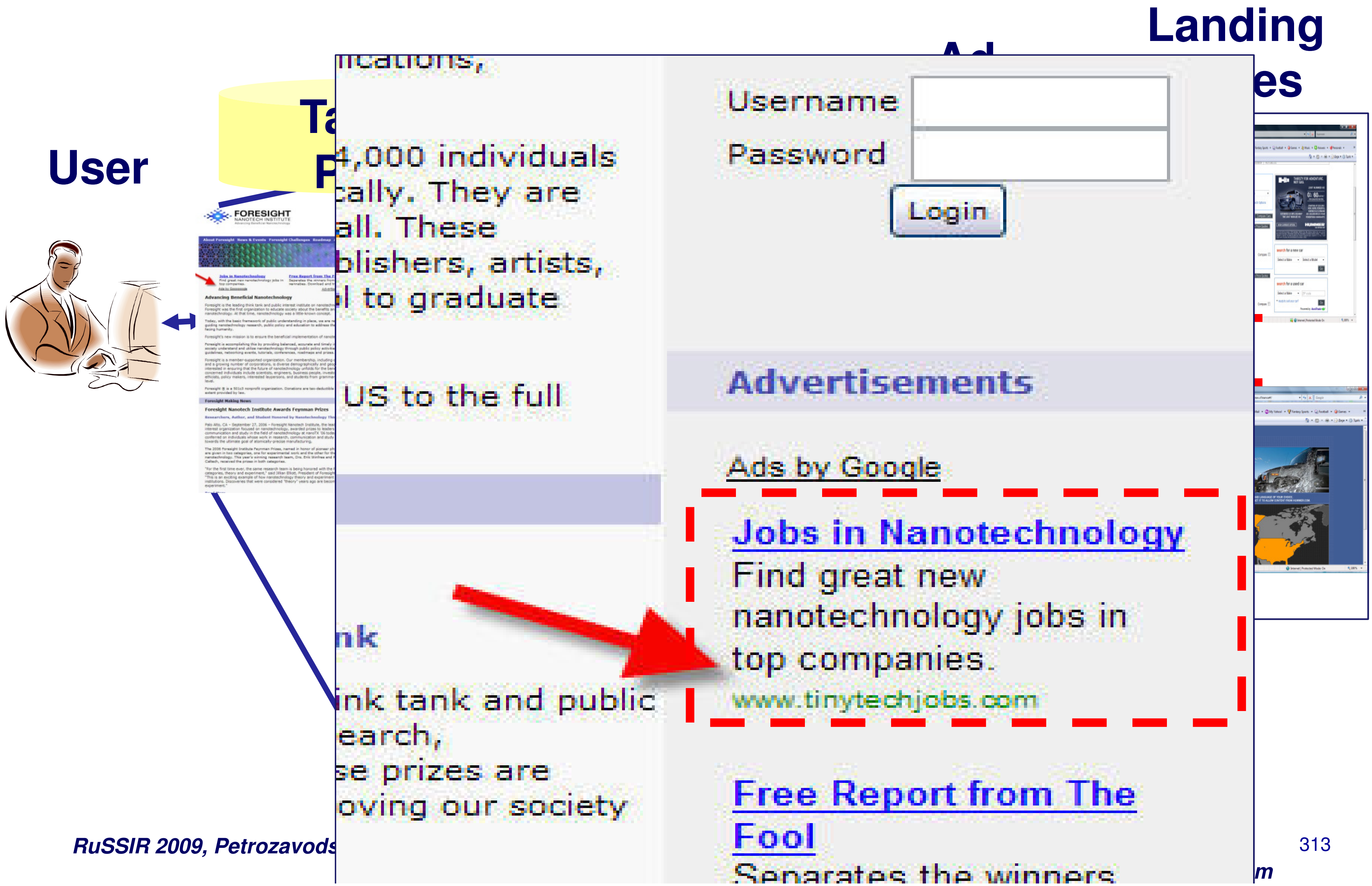
Online Auctions Outline

- **Introduction to Auctions**
- **Game Theory**
  - Matrix games versus strategic form games
  - I.e., **2**-person games versus **N**-person games
- **Finding Equilibria solutions/outcomes in games**
  - Games with a dominant strategy
  - Pure-strategy Nash Equilibrium (NE)
  - Mixed strategy NE
- **Repeat Games (finite and infinite)**
- **Multi-item auctions (VCG, GSP)**
- **Online Ad Auctions**

Pub: How much for an impression?



Establishing the price for an ad slot?

Problem

- **Publisher wishes to sell a ad slot for which there are many interested buyers/advertisers**
  - Versus one buyer trying to buy a single item (procurement auctions).
- **Want to establish a price for the object(ad slot)**
  - If seller knows each potential buyer's value of object (or has a good estimate) then the seller can just announce the price at which the object is sold
  - However, if the seller does not know the buyer's value, and the buyers do not know each others' values for the object (i.e., independent private values) then auctions help
    - Each buyer has an intrinsic value for the item being auctioned; she is willing to purchase the item for a price up to this value
    - Auctions help to discover true valuation

Announce Price

Discover Price Via Auction

Types of Auctions

- **Two main categories**
  - Open Outcry
    - Ascending
    - Descending
  - Sealed Bid
    - First-price
    - Second-price
- **Main idea: bidder trying to balance their private-value with what they are willing to bid (the cost to them) for an item.**

Forward/Futures Markets

ONLINE ADVERTISING RATE SHEET

Chicagoreader.com
More than 100,000 unique users and 1,000,000 pageviews every week

Chicagoreader.com focuses on function, popular features, and daily updates. Our homepage is an essential portal into local arts, entertainment, and issues. *Chicago Reader On Film* archives more than 10,000 capsule movie reviews. The *Reader Restaurant Finder* is an online guide to more than 3,000 area restaurants. *Reader Online Classifieds* are a complete online marketplace for apartment rentals, houses and condos, jobs, personals, services, and more.

Online Ad Rates

50,000 - 199,000 ad impressions	**$12 per 1,000**
200,000 - 499,000 ad impressions	**$10 per 1,000**
500,000 + ad impressions	**$8 per 1,000**

Online Ad Sizes

Leaderboard	Top of page	728 pixels x 90 pixels
Skyscraper	Right hand column	160 pixels x 600 pixels
Rectangle	Within text	300 pixels x 250 pixels

Hybrid Advertising: Print + Online
50% of our print readers use chicagoreader.com. (2006 MRI Survey)

Advertisers can increase the reach and frequency of their print advertising with simultaneous ad impressions on chicagoreader.com. Reach our total audience with the combination of the Chicago Reader and chicagoreader.com.

High human touch
-Use rate cards
-Auction/negotiate

Keyword Auction Systems: Goto Model

- **Rank ads by keyword bid price**
  – each ad is associated with multiple keywords; assume one keyword for now and exact match
- **In 1997, Goto/Overture (now Yahoo! Search Marketing) launched an innovative framework for selling advertising space next to search results.**
  – Rather than selling large, expensive chunks of advertising space (human sales force), each keyword was sold via its own auction
  – Human editors checked for relevance
  – Payment was made on a pay-per-click (PPC)
  – Used a **first price auction mechanism** (and published the winning bids!!)
  – Successful; advertising system adapted by Yahoo and MSN

Generalized first-price auction (GFP)

- **For each keyword, several advertising slots are auctioned at once, each one representing a position relative to the top of the search page.**
- **Overture created a marketplace around each keyword**
  – Their auction mechanism has been characterized as a generalized first-price auction (GFP) .
  – Each advertiser submits a ***secret bid*** (value of click/action) to the auctioneer (Overture in this case).

1st Price – In a first-price auction for a ***single item***, the highest bidder wins the item at the highest price.

GFP – In a **GFP**, ***multiple items*** are up for auction; the highest bidder wins the first item at the highest price, the second-highest bidder wins the second item at the second-highest price, and so on.

Generalized First Price Auction

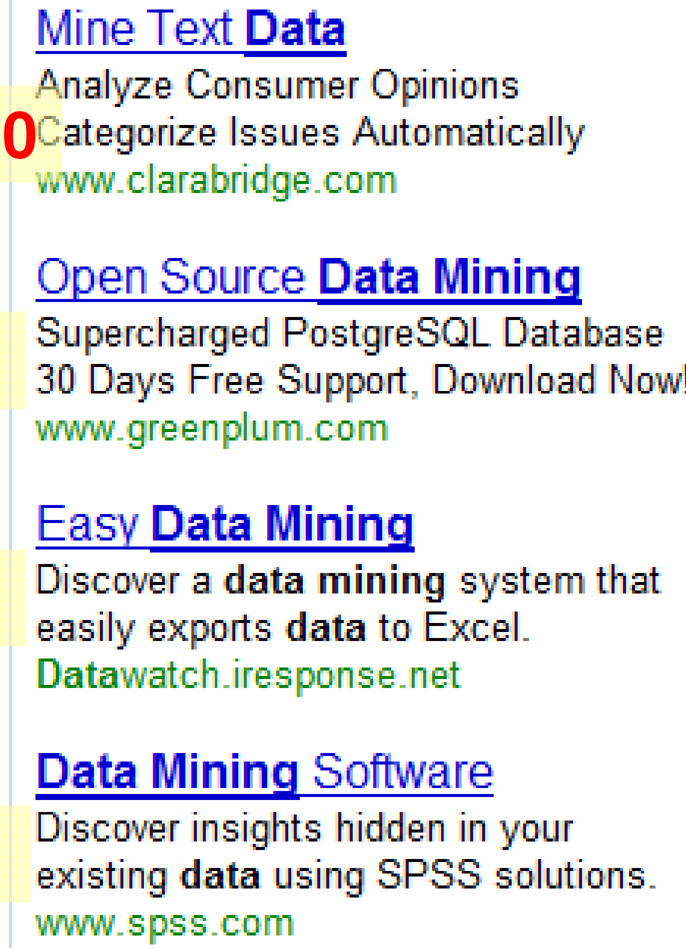
1. In a GFP, multiple items are up for auction;
2. The highest bidder wins the first item at the highest price
3. The second-highest bidder wins the second item at the second-highest price, and so on

KW Bid = $10

KW Bid = $5

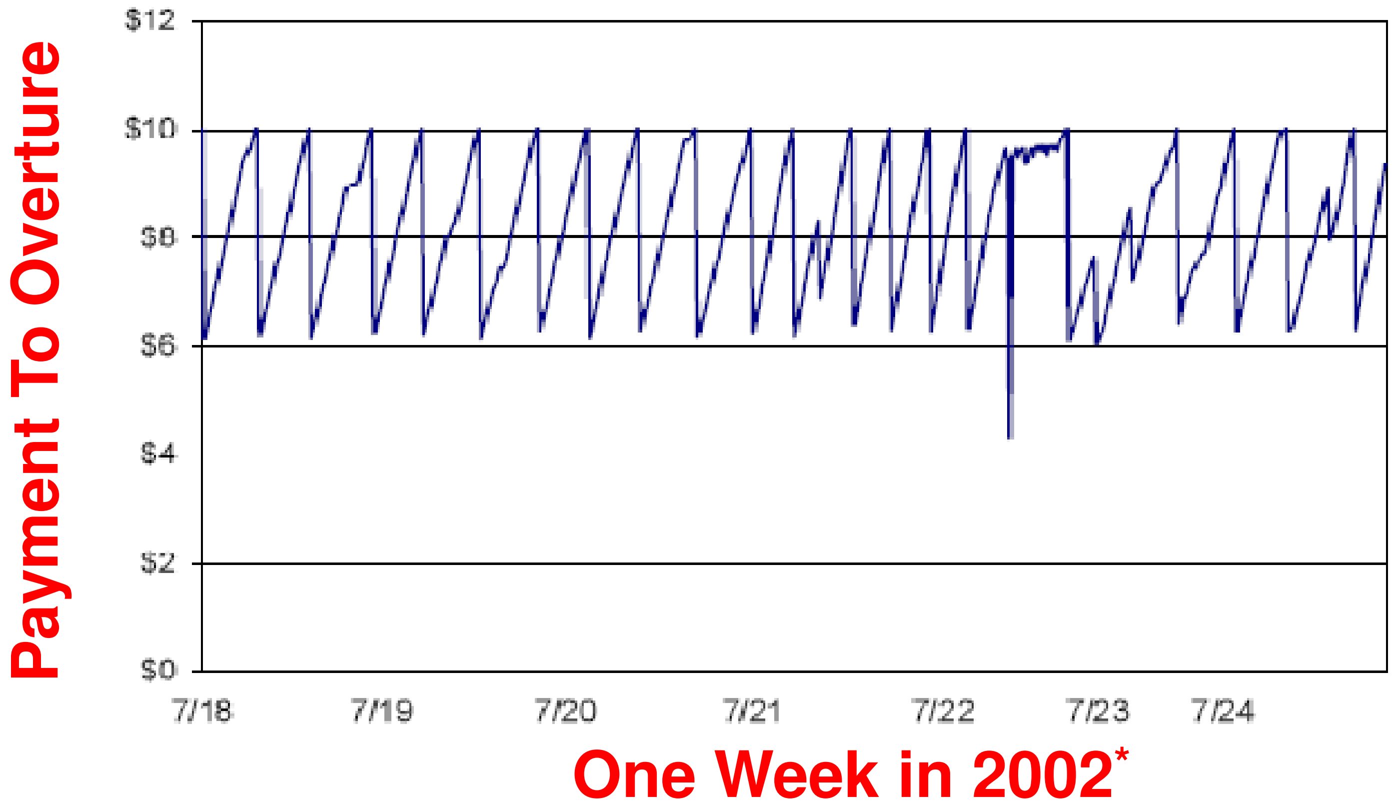
KW Bid = $2

KW Bid = $1



Gaming the system: GFP not stable

- **Another notable aspect of Overture's auction design was that winning bids were posted**
- **Led to buyer’s remorse and gaming systems; no equilibrium**



***[Edelman, B. et al.**Internet advertising and the generalized second price auction: selling billions of dollars worth of keywords**. NBER Paper No. W11765, 2005]**

Generalized 2nd Price (GSP) Auction

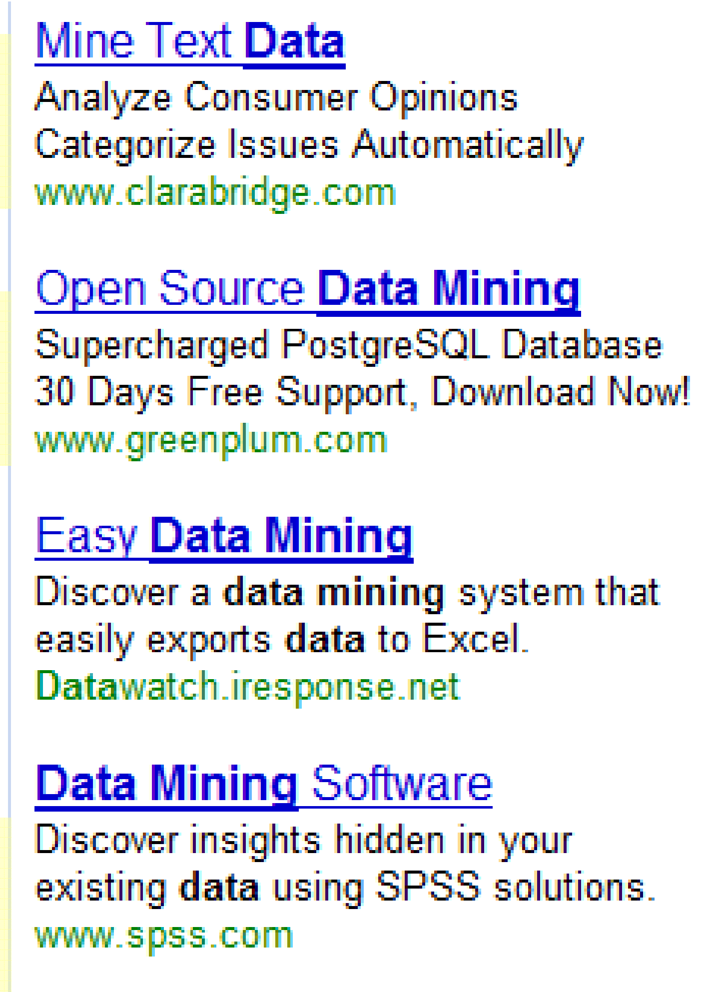
1. In a GSP, multiple items are up for auction;
2. The highest bidder wins the first item at the second price (+delta)
3. The second-highest bidder wins the second item at the third-highest price, and so on

Bid = $10
PPC = $5

Bid = $5
PPC = $2

Bid = $2
PPC = $1

Bid = $1
PPC = $0.57

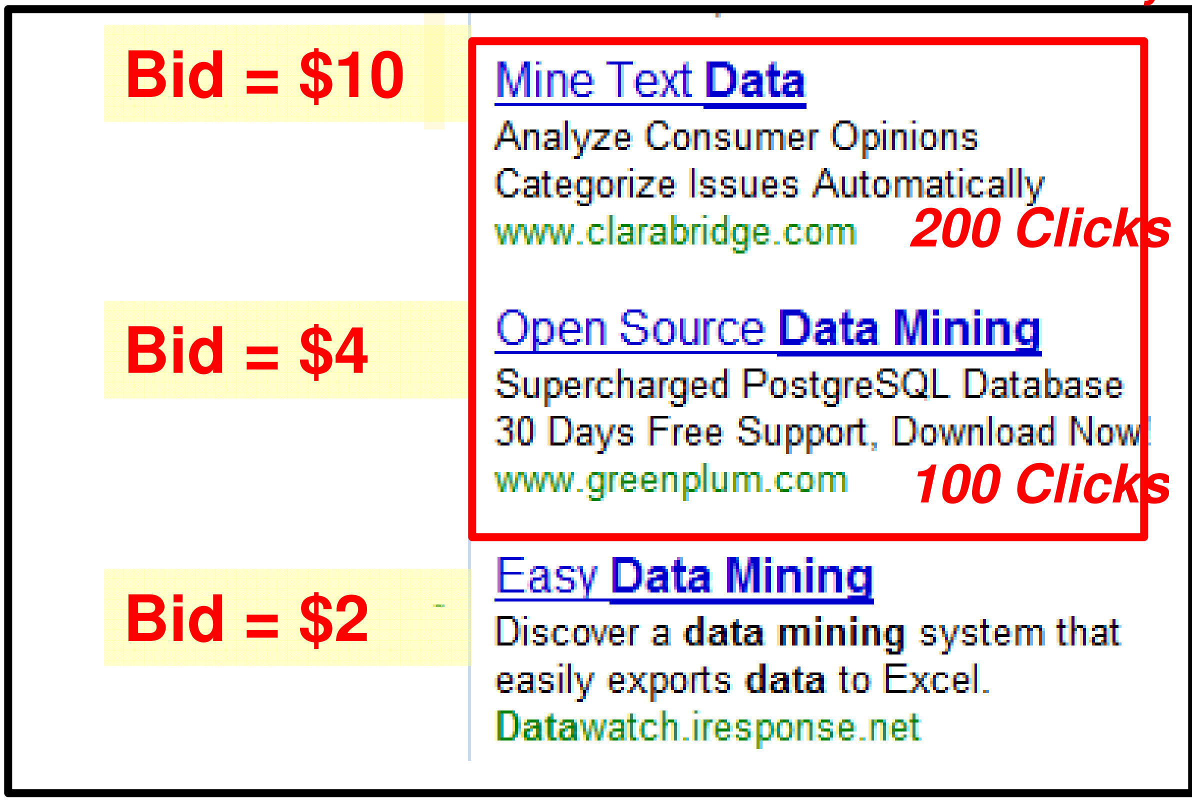


Introduced by Google in Feb 2002 (AdWords); overcomes the instability of GFP because by design the bidder is incentivized to pay the true value?!

Example Auction

Assume 2 ads slots only

Note:
However, in a GSP/VCG auction, advertisers must submit a single bid even though there are several advertisement slots available.



Suppose there are two slots on a page and three advertisers. An ad in the first slot receives 200 clicks per hour, while the second slot gets 100.

Generalized 2nd Price (GSP) Auction

Assume 2 ads slots only

1. In a GSP, multiple items are up for auction;
2. The highest bidder wins the first item at the second price (+delta)
3. The second-highest bidder wins the second item at the third-highest price, and so on

Bid = $10
PPC = $4
Payment =$4*200

Mine Text **Data**
Analyze Consumer Opinions
Categorize Issues Automatically
www.clarabridge.com *200 Clicks*

Bid = $4
PPC = $2
Payment =$2*100

Open Source **Data Mining**
Supercharged PostgreSQL Database
30 Days Free Support, Download Now!
www.greenplum.com *100 Clicks*

Bid = $2
PPC = $2

Easy **Data Mining**
Discover a **data mining** system that easily exports **data** to Excel.
Datawatch.iresponse.net

Revenues under GSP is $1,000

Online Auctions Outline

- **Introduction to Auctions**
- **Game Theory**
  - Matrix games versus strategic form games
  - I.e., **2**-person games versus **N**-person games
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  - Games with a dominant strategy
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  - Mixed strategy NE
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Establishing the price for an ad slot?

Problem

- **Publisher wishes to sell a ad slot for which there are many interested buyers/advertisers**
  - Versus one buyer trying to buy a single item (procurement auctions).
- **Want to establish a price for the object(ad slot)**

Announce Price

- – If seller knows each potential buyer's value of object (or has a good estimate) then the seller can just announce the price at which the object is sold

Discover Price Via Auction

- – However, if the seller does not know the buyer's value, and the buyers do not know each others' values for the object (i.e., independent private values) then auctions help
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Types of Auctions

- **Two main categories**
  - Open Outcry
    - Ascending
    - Descending
  - Sealed Bid
    - First-price
    - Second-price
- **Main idea: bidder trying to balance their private-value with what they are willing to bid (the cost to them) for an item.**

Types of Auctions: Open Outcry

- **Basic idea: bids are public and thus made public knowledge.**

Two Types

- **Descending (aka Dutch): (only one bid)**
  - Auctioneer starts at high price and decreases.
  - Winner: agent that stops the auctioneer and accepts the price.
  - Analogous to sealed-bid first price auction
- **Ascending (aka British): (possible multiple bids)**
  - Auctioneer starts at low price and price increases as bidders increase bids.
  - Winner: agent with highest bid when no more bids occur.
  - Analogous to sealed-bid second price auction

Types of Auctions : Sealed Bid

- **Basic Idea: bids are private and made public only upon announcement of winner.**

Two Types

- **First Price (only one bid)**
  - Bidder with the highest bid wins
  - Pays the amount of their bid.
- **Second Price (only one bid) (aka Vickrey Auctions)**
  - Bidder with the highest bid wins
  - But only pays the amount for the second highest bid.

Types of Auctions : Valuation

- **Common/Objective Value**
  - There is a value shared by all bidders for an item.
  - Value may be imprecise:
    - Individual agents may have their assessor’s prediction of the value of something which may be different to another’s.
  - eg. Vein of some mineral will have common value to all mining companies.
- **Private/Subjective Value**
  - Bidders place different values on objects.
  - Bidder’s know private valuations but not others’.
  - Seller does not know valuations.
  - Depending on auction structure – agents may be able to formulate idea of valuations from bidding signals.

The Winner's Curse

- **Definition: if you have won an auction, you may have overpaid.**
  - Mostly for common value auctions
- **Propose a bid *b***
- **Win the bid if current owner's valuation is between [0; *b*]**
- **If you control item it's worth is 1.5*b***
- **But *average* value for the item would be *b/2* if a bid you make is accepted.**
- **Thus under your control it's worth:**

  1.5(*b/2*) = 0.75*b*
- **This means that whatever you pay it will always be worth less than what you chose to bid!**

Good Bidding Strategies

- **Ascending**
  - Item worth *V* to you.
  - If last bid above *V* there is no reason to bid.
  - If last bid is *r,* below *V* - then you bid *r* plus the minimum bid increment ε (epsilon – a small amount).
  - This means that your profit approximates: *V-r*
  - This is approximately the second price.
- **First-Price Sealed**
  - Need to ***shade*** your bid in order to make a profit.
    - ***Shading***: is when you place a bid less than your value, *V* but not so low as to guarantee losing. Involves risk for reward.
- **Descending**
  - Similar to FPS, you need to ***shade*** your bid.

2nd Price (Vickrey) Auctions

- **Truthful bidding is a dominant strategy!**
- **Some item worth *V* to you.**
- **You can place any bid *b*, *b* can be any positive number.**
- **If you don't bid *b=V* there are two possibilities:**
  - Opponent bids higher than you.
  - Opponent bids lower than you.
- **We show that for each of these it's better for you to bid *b=V* rather than $b \neq V$.**

2^{nd} Price (Vickrey) Auctions 2

- **Let opponent's bid =*r***
- **Opponent Bids Higher:**
  - v < b < r: opponent wins, wouldn't change anything if b=v.
  - b < v < r: opponent wins, wouldn't change anything if b=v
  - b < r < v: opponent wins, would be strictly better if b=v (you would have won)
- **Opponent bids lower**
  - v < r < b: you win, but you now pay some amount and net v-r < 0.
  - r < v < b: you win, wouldn't have changed if v=b.
  - r < b < v: you wing, wouldn't have changed if v = b.

Online Auctions Outline

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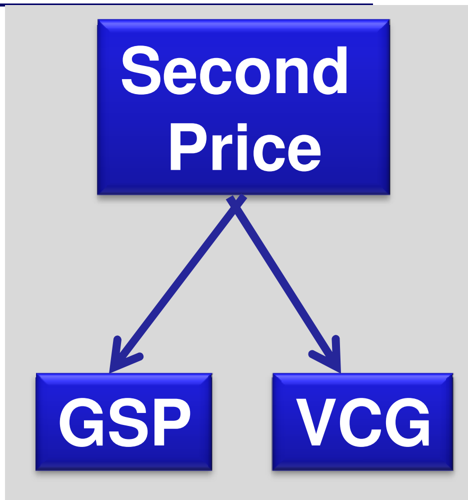
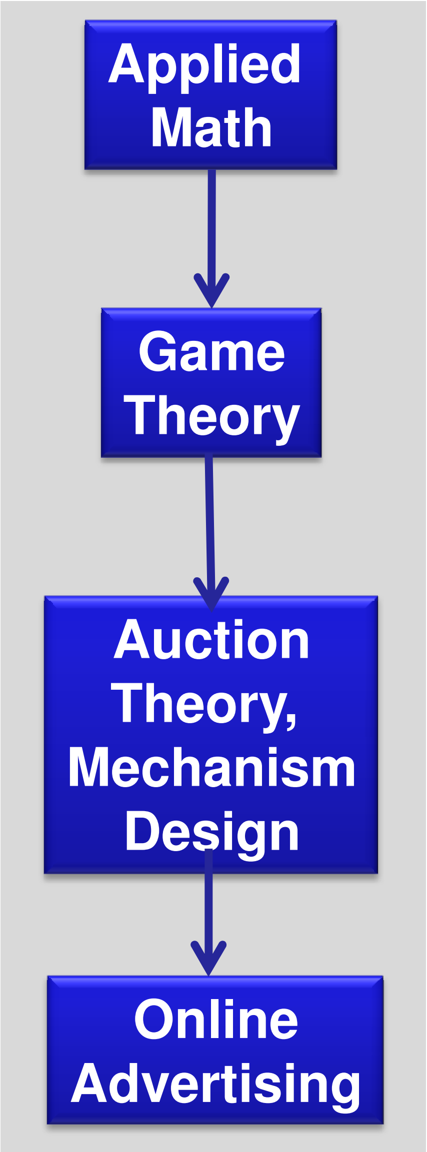
First price (GFP) vs. Second Price GSP



- **Generalized First Price Auction**
  - Unstable
- **Second Price Auction (Single Item)**
  - Truth-telling is the dominant strategy
  - (i.e., no buyer's remorse when bidding true value)
- **Generalized 2nd Price (GSP) Auction**
  - Tailored to the unique environment of online ads [Google, 2002]
  - BUT truth-telling is NOT a dominant strategy for Generalized Second Price (GSP) Auctions [Edelman et al. 2006]
- **Vickrey, Clarke, Groves (VCG) Auction**
  - Truth-telling is a dominant strategy under VCG
  - In particular, unlike the VCG mechanism, GSP generally does not have an equilibrium in dominant strategies and truth-telling is not an equilibrium of GSP.

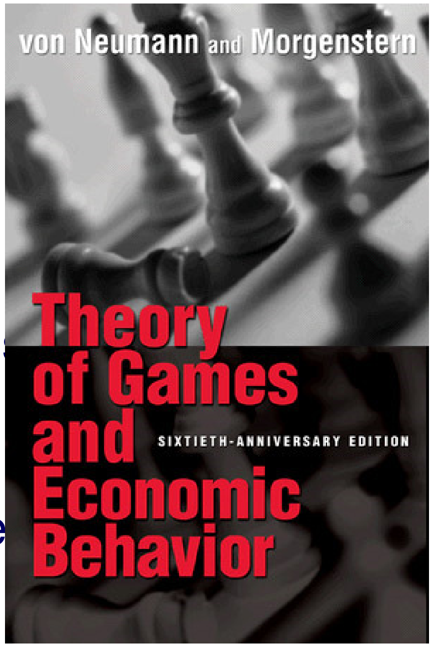
Auction Theory:a branch of Game Theory

- **Game theory is a branch of applied math**
  - that is used in the social sciences (most notably economics), biology, engineering, political science, international relations, computer science, philosophy.
- **Game theory is the science of strategy**
  - It attempts to determine mathematically and logically the actions that "players" should take to secure the best outcomes for themselves in a wide array of "games."
- **Auction theory is a branch of game theory**
- **An online advertising auction is a game:**
  - the strategies/bids of all participants determine both the winner and the winning price
  - Game theory provides a formal means of understanding and designing auctions



Game Theory Background

- **Developed in the late 1920s, game theory is concerned with the decisions people make when confronted with competitive situations**
  - Especially when they have limited information about the other players' choices
  - It attempts to determine mathematically and logically the actions that "players" should take to secure the best outcomes for themselves in a wide array of "games."
  - The decisions of all agents jointly determine the game outcome



- **Every competitive situation has a point called a Nash Equilibrium, in which parties cannot change their course of action without sabotaging themselves**
  - Every finite player, finite strategy game has at least one Nash equilibrium be it a mixed or pure strategy equilibria [Nash 1950]; (Proof is based on Kakutani's fix point theorem)
  - Nash got a Nobel Prize for this
  - In 1838 Cournot considers a duopoly and presents a solution that is a restricted version of the Nash equilibrium

[http://en.wikipedia.org/wiki/Game_theory]

Game Theory Outline

- **Matrix games versus strategic form games**
  - I.e., **2**-person games versus **N**-person games
- **Finding Equilibria solutions/outcomes in games**
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  - Pure-strategy Nash Equilibrium (NE)
  - Mixed strategy NE
- **Repeat Games (finite and infinite)**
- **Truth-telling**
- **Online Ad Auctions**

Matrix Game

- **Players, strategies and payoffs**
- **A matrix game is a two player game such that:**
  - player 1 has a finite strategy set S1 with m elements,
  - player 2 has a finite strategy set S2 with n elements, and
  - the payoffs of the players are functions u1(s1, s2) and u2(s1, s2) of the outcomes (s1, s2) $\in$ S1 $\times$ S2.
- **The matrix game is played as follows:**
  - at a certain time player 1 chooses a strategy s1 $\in$ S1 and simultaneously player 2 chooses a strategy s2 $\in$ S2 and once this is done each player i receives the payoff u_i(s1, s2).

A matrix game: Prisoner's Dilemma

Suspect 2 Strategy

Suspect 1 Strategy

	Rat Out	Stay Quiet
Rat Out	-5 -5	-10 0
Stay Quiet	0 -10	-2 -2

- **2 strategies per player (Suspect Rat-out or stay quiet)**
- **Payoffs (cell entries) are a function of the strategies selected by each player (simultaneously)**
  - If suspect1 stays quiet and suspect 2 rats out then suspect 1 gets 10 years in prison (looses 10 years) while suspect 2 receives zero years

Some Notation and Definitions

- **Strategy choices for all player besides player *i***
  - $s_{-I} = (\ldots, s_{i-1}, s_{i+1}, \ldots.)$
- **Strategy s_i^* is a <u>Best Response</u> by player *i* to the strategies of all players except *i*, s_{-i} if:**

$$\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i, s_{-i}) \text{ for all } s_i \in S_i$$

$$\pi_1(RatOut, S2) \geq \pi_2(StayQuiet, S2) \text{ where } S2 \in \{RatOut, StayQuiet\}$$

- **Strategy s_i^* is a <u>Dominant Strategy</u> for player *i* if s_i^* is a best response**

	Suspect1\Suspect2	Rat Out	Stay Quiet
s_1^*	Rat Out	-5, -5	0, -10
s_1	Stay Quiet	-10, 0	-2, -2

Dominant Strategies

Suspect1\Suspect2	Rat Out	Stay Quiet
Rat Out	-5 / -5	-10 / 0
Stay Quiet	0 / -10	-2 / -2

- **For two-person matrix games, a strategy s_i of player 1 in a matrix game is said to dominate another strategy s_j of player 1 if**
  - $u_1(s_i, s) \geq u_1(s_j, s)$ //payoff of player 1
- **For suspect 1 ratOut dominates stayQuiet**
  - u1(ratOut, $S2) ≥ u1(stayQuiet, $S2)
  - u1(ratOut, ratOut) ≥ u1(stayQuiet, ratOut) AND u1(ratOut, stayQuiet) ≥ u1(stayQuiet, stayQuiet)
  - -5 ≥ -10 AND 0 ≥ -2

A strategy s_i of player1 that dominates another strategy s_j gives player1 a higher payoff for every choice that player 2 could make

Strictly Dominant Strategies

Suspect 1 Strategy

	Rat Out	Stay Quiet
Rat Out	-5 -5	-10 0
Stay Quiet	0 -10	-2 -2

- **For two-person matrix games, a strategy s_i of player 1 in a matrix game is said to <u>dominate</u> another strategy s_j of player 1 if**
  - u1(si, s) **>** u1(sj , s)   #payoff of player 1
- **For suspect 1 ratOut dominates stayQuiet**
  - u1(ratOut, $S) > u1(stayQuiet, $S)
  - u1(ratOut, ratOut)>u1(stayQuiet, ratOut)
    AND u1(ratOut, stayQuiet) > u1(stayQuiet, stayQuiet)
  - -5 > -10 AND 0 > -2

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Iterated Elimination of Dominated Strategies

- **Find a solution to the game by iteratively eliminating strictly dominated strategies**
  - Let $Ri \in Si$ be the set of removed strategies for agent i
  - Initially $Ri = \emptyset$
  - Choose agent i, and strategy si such that $si \in Si \setminus Ri$ and there exists $si' \in Si \setminus Ri$ such that

$$u_i(si', s_{-i}) > u_i(si, s_{-i}) \text{ for all } s_{-i} \in S_{-i} \setminus R_{-i}$$

  - Add si to Ri, continue

- **Theorem: If a unique strategy profile, s*, survives then it is a Nash Equilibrium**
- **Theorem: If a profile, s*, is a Nash Equilibrium then it must survive iterated elimination.**

Example: Iterated Dominance

- Players iteratively throw out strictly dominated strategies
- → leads to a solution of a matrix game

	r	l	c
U	3,-10	7,-7	9,-15
D	9,-9	8,-8	10,-10

Prisoner's Dilemma Game

Suspect 2 Strategy

Suspect 1 Strategy

	Rat Out	Stay Quiet
Rat Out	-5 / -5	-10 / 0
Stay Quiet	0 / -10	-2 / -2

- **Two suspects**
  - Develop a system where the suspects will want to admit their crime
  - Separate rooms: No cooperation; prevent them from colluding
- **Dilemma: rational players are expected to play their dominant strategies (better payoffs), whereas a more optimal outcome exists**
  - If suspect1 ratsOut he gets a better payoff for each choice that suspect2 makes (and similarly for suspect2); in the absence of any communication, rational players are expected to play their dominant strategies, since a strictly dominant strategy gives a player an unequivocally higher payoff
  - The solution using strictly dominant strategies will give each suspect 5 years, which, of course, is a worse outcome than if each suspect could trust the other to stay quiet

Prisoner's Dilemma Theorem

- **If Prisoner's dilemma is played by rational players, both players confess.**
- **Proof:**
  - "StayQuiet" is a **strongly dominated** strategy:
  - Strongly dominated = Its payoff is **strictly less** than the payoff of "RatOut" for all states of the world.
  - Therefore, "StayQuiet" is ***not*** a best reply, irrespective of belief.

Iterated Dominance Alg. Limitations

Battle of the sexes

Female Player

Male Player

Strategy	Opera	Bullfight
Opera	2 / 1	0 / 0
Bullfight	0 / 0	1 / 2

- **The female prefers opera to bullfight , while the male prefers bullfighting to opera; but they also want to spend time together!**
- **This game has no strictly dominating strategies ☹**
- **How can we determine a solution to this game?**
  - Nash's Equilibrium provides us with a solution

Find solutions to a game

- **The main tool is to find an *equilibrium*: a set of choices by all agents that are mutually rational**

Nash's Equilibrium for a Matrix Game

- **A Nash equilibrium is a widely used method of predicting the outcome of a strategic interaction such as an games/online auctions.**
- **A pair of strategies $(s_1^*, s_2^*) \in S1 \times S2$ is a Nash equilibrium of a matrix game if:**
  – 1. $u_1(s_1^*, s_2^*) \geq u_1(s, s_2^*)$ for each $s \in S1$, and
  – 2. $u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s)$ for each $s \in S2$

- **In other words, a Nash equilibrium is an outcome (i.e., a pair of strategies) of the game from which none of the players have an incentive to deviate, as, given what the other player is doing, it is optimal for a player to play the Nash equilibrium strategy.**

Nash's Equilibrium

Player2			
	Strategy	L	R
Player1	T	1, 1	0, 0
	B	0, 0	1, 1

- **This game does not have strictly dominated strategies BUT has two Nash equilibria, namely (T,L) and (B,R).**
- **It is also worth noting that if we look at an outcome which is not a Nash equilibrium then one player will want to deviate from playing that outcome.**
  - E.g., for strategy (T,R), then player 2 is better-off playing L if he knows that player 1 is going to play T.
- **Nash's Equilibia $\supset$ Equilibia found by iterative dominated strategies**
- ***Games may also have mixed strategies***

[John Nash,1951]

Pareto-optimal Equilibrium Points

- **A game outcome is Pareto-optimal if there is no other outcome that all players would prefer.**
- **An outcome is Pareto-dominated by another outcome if all players would prefer the other outcome**

	Player2		
Player1	Strategy	L	R
	T	9 / 9	0 / 0
	B	0 / 0	1 / 1

	Player2		
Player1	Strategy	L	R
	T	1 / 1	0 / 0
	B	0 / 0	1 / 1

- **Two equal PoEPs**
- **Requires Communication**
- **Establish convention before or during**
- **Coordination Game**

Inefficient Equilibrium

Airline Price Fixing Lawsuit: Game Theory

- **US Justice Department settled an US$billion antitrust suit against six major airlines**
  - The airlines were accused of using the computerized system to negotiate future fares with competitors. Some future fares were placed in the computerized system two months in advance, but most took effect within a week or two.
  - A Justice Department spokesman called the system "an electronic smoke-filled room" used by airlines to float "trial balloon" price increases, make and receive counterproposals and reach a consensus on the amount and timing of price increases or the removal of discounts.
  - *The settlement prohibits the airlines from announcing future fares. Under the agreement, airline fares must be available when they are announced. The settlement also prohibits the announcement of the last day on which a discount can be offered.*
- **Setting the fare of $200 is a strictly dominant strategy for both airlines (in our example). Hence, the strictly dominant strategy solution causes both airlines to make a loss of $10 million.**
- **This then provides airlines with an incentive to try and reach some form of a price fixing agreement.**

[NYTime, March 18, TOLCHIN, 1994]

Airline Price Fixing Lawsuit

Example 2.1. Suppose US Air and American Airlines (AA) are thinking about pricing a round trip airfare from Chicago to New York. If both airlines charge a price of $500, the profit of US Air would be $50 million and the profit of AA would be $100 million. If US Air charges $500 and AA charges $200 then the profit of AA is $200 million and US Air makes a loss of $100 million. If, however, US air sets a price of $200 and AA charges $500 then US Air makes a profit of $150 million while AA loses $200 million. If both charge a price of $200 then both airlines end up with losses of $10 million each. This information can be depicted in the form of a table as shown below. ■

	American Airlines		
US Air	Fare	$500	$200
	$500	(50,100)	(−100,200)
	$200	(150,−200)	(−10,−10)

Loose $10M each

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

Strategic From Games (n-player)

- **Extend games to n-players and to player strategy sets that do not have a nice matrix representation.**
- **A strategic form game (or a game in normal form) is simply a set of *n* players labeled** *1, 2, . . . , n* **such that each player *i* has:**
  - 1. a choice set S_i (also known as the strategy set of player *i* and its elements are called the strategies of player *i*), and
  - 2. a payoff function $u_i: S_1 \times S_2 \times \cdots \times S_n \rightarrow \Re$.
- **The game is played as follows:**
  - each player *k* chooses simultaneously a strategy $s_k \in S_k$ and once this is done each player *i* receives the payoff $u_i(s_1, s_2, . . . , s_n)$.
  - Represent a game in terms of strategy sets and payoff functions of the players: $G = \{S_1, . . . , S_n; u_1, . . . , u_n\}$

Nash's Eqlbrm. for Strategic Form Game

- **When a strategic form game is played, a player's objective is to maximize her payoff.**
  - (all other players will want to do the same.).
- **Look for an game outcome that results from the simultaneous maximization of individual payoffs**
- **There is a useful criterion for finding the Nash equilibrium of a strategic form game when the strategy sets are open intervals of real numbers.**

Definition 2.8. *A* Nash equilibrium *of a strategic form game*

$$G = \{S_1, \ldots, S_n, u_i, \ldots, u_n\}$$

is a strategy profile $(s_1^*, s_2^*, \ldots, s_n^*)$ *such that for each player* i *we have*

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s, s_{i+1}^*, \ldots, s_n^*)$$

for all $s \in S_i$.

No incentive to deviate from their chosen strategy

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Nash Equilibrium Test

- **There is a useful criterion for finding NE of a strategic game when strategy sets are open intervals of real numbers payoff functions are twice differentiable**

Let G be a strategic form game whose strategy sets are open intervals and with twice differentiable payoff functions. Assume that a strategy profile $(s_1^*, \ldots, s_2^*)$ *satisfies:*

1. $\dfrac{\partial u_i(s_1^*, \ldots, s_n^*)}{\partial s_i} = 0$ *for each player i,*

2. *each* s_i^* *is the only stationary point of the function*

$$u_i(s_1^*, \ldots, s_{i-1}^*, s, s_{i+1}, \ldots, s_n^*)\,, \quad s \in S_i\,,$$

and

3. $\dfrac{\partial^2 u_i(s_1^*, \ldots, s_n^*)}{\partial^2 s_i} < 0$ *for each i.*

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

Then $(s_1^*, \ldots, s_n^*)$ *is a Nash equlibrium of the game G.*

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John F Nash Jr. (1928 -)

Landmark contributions to Game theory: notions of Nash Equilibrium and Nash Bargaining

Nobel Prize : 1994

***A Beautiful Mind*, about his mathematical genius and his struggles with schizophrenia**

Find Nash Eql.

Example 2.9. Consider a three-person strategic form game in which each player has a strategy set equal to the open interval $(0, \infty)$. The payoff functions of the players are given by

$$\begin{aligned} u_1(x,y,z) &= 2xz - x^2 y \\ u_2(x,y,z) &= \sqrt{12(x+y+z)} - y \\ u_3(x,y,z) &= 2z - xyz^2 \,. \end{aligned}$$

Assume payoff functions are twice differentiable

To find the Nash equilibrium of the game, we must solve the system of equations

$$\frac{\partial u_1}{\partial x} = 0, \quad \frac{\partial u_2}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u_3}{\partial z} = 0\,.$$

Taking derivatives, we get

$$\frac{\partial u_1}{\partial x} = 2z - 2xy, \quad \frac{\partial u_2}{\partial y} = \sqrt{\frac{3}{x+y+z}} - 1 \quad \text{and} \quad \frac{\partial u_3}{\partial z} = 2 - 2xyz\,.$$

So, we must solve the system of equations

$$2z - 2xy = 0, \quad \sqrt{\frac{3}{x+y+z}} - 1 = 0 \quad \text{and} \quad 2 - 2xyz = 0\,,$$

or, by simplifying the equations,

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

So, we must solve the system of equations

$$2z - 2xy = 0, \quad \sqrt{\frac{3}{x+y+z}} - 1 = 0 \quad \text{and} \quad 2 - 2xyz = 0\,,$$

or, by simplifying the equations,

$$z = xy \qquad (1)$$
$$x + y + z = 3 \qquad (2)$$
$$xyz = 1\,. \qquad (3)$$

Substituting the value of xy from (1) to (3) yields $z^2 = 1$, and (in view of $z > 0$) we get $z = 1$. Now substituting the value $z = 1$ in (1) and (2), we get the system

$$xy = 1 \quad \text{and} \quad x + y = 2\,.$$

Solving this system, we obtain $x = y = 1$. Thus, the only solution of the system of equations (1), (2) and (3) is $x = y = z = 1$.

Computing the second derivatives, we get

$$\frac{\partial^2 u_1}{\partial x^2} = -2y < 0\,,$$
$$\frac{\partial^2 u_2}{\partial y^2} = -\frac{\sqrt{3}}{2}(x+y+z)^{-\frac{3}{2}} < 0\,,$$
$$\frac{\partial^2 u_3}{\partial z^2} = -2xy < 0\,,$$

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

for all choices $x > 0$, $y > 0$ and $z > 0$. The Nash Equilibrium Test guarantees that $(1, 1, 1)$ is the only Nash equilibrium of the game. ∎

A Mixed Strategy Nash Eqlm.

- **A mixed strategy or probability profile for the row player is simply any vector *$p = (p_1, p_2, \ldots, p_m)$* such that *$p_i \geq 0$* for each strategy *$i$* and *$\sum_{i=1..n} p_i = 1$*.**
- **A player picks a distribution and not just one strategy**
- **Pure Strategy**
  - A mixed strategy p for the row player is said to be a pure strategy, if for some strategy i we have $p_i = 1$ and $p_k = 0$ for $k \neq i$. E.g., $p = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$
- **While the game might not have an equilibrium in pure strategies, it always has a mixed strategy equilibrium!**

Finding a Mixed Strategy Equilibria

To compute mixed strategies equilibria in a matrix game we use the following four steps.

1. Write the matrix game in its bimatrix form $A = [a_{ij}]$, $B = [b_{ij}]$.
2. Compute the two payoff functions **Expected Payoff Functions**

$$\pi_1(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j a_{ij} \quad \text{and} \quad \pi_2(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j b_{ij}\,.$$

3. Replace $p_m = 1-\sum_{i=1}^{m-1} p_i$ and $q_n = 1-\sum_{j=1}^{n-1} q_j$ in the payoff formulas and express (after the computations) the payoff functions π_1 and π_2 as functions of the variables $p_1,\dots,p_{m-1},q_1,\dots,q_{n-1}$.
4. Compute the partial derivatives $\frac{\partial \pi_1}{\partial p_i}$ and $\frac{\partial \pi_2}{\partial q_j}$ and consider the system

$$\frac{\partial \pi_1}{\partial p_i} = 0 \ (i = 1,\dots,m-1) \quad \text{and} \quad \frac{\partial \pi_2}{\partial q_j} = 0 \ (j = 1,\dots,n-1)\,.$$

Any solution of the above system $p_1,\dots,p_{m-1},q_1,\dots,q_{n-1}$ with

$$p_i \geq 0 \text{ and } q_j \geq 0 \text{ for all } i \text{ and } j, \quad \sum_{i=1}^{m-1} p_i \leq 1 \quad \text{and} \quad \sum_{j=1}^{n-1} q_j \leq 1$$

is a mixed strategies equilibrium.

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

Find Mixed Strategy Equilibria: E.g.

To compute mixed strategies equilibria in a matrix game we use the following four steps.

1. Write the matrix game in its bimatrix form $A = [a_{ij}]$, $B = [b_{ij}]$.

2. Compute the two payoff functions

$$\pi_1(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j a_{ij} \quad \text{and} \quad \pi_2(\mathbf{p},\mathbf{q}) = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j b_{ij}\,.$$

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4. Compute the partial derivatives $\frac{\partial \pi_1}{\partial p_i}$ and $\frac{\partial \pi_2}{\partial q_j}$ and consider the system

$$\frac{\partial \pi_1}{\partial p_i} = 0 \ (i = 1, \ldots, m-1) \quad \text{and} \quad \frac{\partial \pi_2}{\partial q_j} = 0 \ (j = 1, \ldots, n-1)\,.$$

Any solution of the above system $p_1, \ldots, p_{m-1}, q_1, \ldots, q_{n-1}$ with

$$p_i \geq 0 \text{ and } q_j \geq 0 \text{ for all } i \text{ and } j, \quad \sum_{i=1}^{m-1} p_i \leq 1 \quad \text{and} \quad \sum_{j=1}^{n-1} q_j \leq 1$$

is a mixed strategies equilibrium.

[Adapted from Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

1. q_1, q_2; p_1, p_2

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix},$$

Expected Payoff Functions

2.

Player A

$$\begin{aligned}\pi_1 &= 3p_1q_2 + 2p_2q_1 + p_2q_2 \\ &= 3p_1(1-q_1) + 2(1-p_1)q_1 + (1-p_1)(1-q_1) \\ &= -4p_1q_1 + 2p_1 + q_1 + 1\end{aligned}$$

Player B

$$\begin{aligned}\pi_2 &= 3p_1q_1 + p_2q_1 + 2p_2q_2 \\ &= 3p_1q_1 + (1-p_1)q_1 + 2(1-p_1)(1-q_1) \\ &= 4p_1q_1 - q_1 - 2p_1 + 2\,.\end{aligned}$$

4.

$$\frac{\partial \pi_1}{\partial p_1} = -4q_1 + 2 = 0 \quad \text{and} \quad \frac{\partial \pi_2}{\partial q_1} = 4p_1 - 1 = 0$$

p_1, p_2 q_1, q_2

$((\frac{1}{4}, \frac{3}{4}), (\frac{1}{2}, \frac{1}{2}))$ is a mixed strategies

Solving Matrix Games using Mixed Strategies

- **Mixed strategy equilibrium**
  - E.g., Rock-Scissors-Paper (RSP) Game (a Zero-sum game)
- **What is the mixed strategy equilibrium for RSP?**
- ?
  - Maximize and solve systems of equations of the expected payoff functions of the players (e.g., δ(p1q2- p1q3 – p2q1 +p2q3 +p3q1 –p3q1)/ δ(p1) and δ(p1q2- p1q3 – p2q1 + p2q3 +p3q1 –p3q1)/ δ(p2) etc.. Solve for p1, p2, p3

Homework

		Player2 (q)		
	Strategy	Rock	Scissors	Paper
Player1 (p)	Rock	0 / 0	-1 / 1	1 / -1
	Scissors	1 / -1	0 / 0	-1 / 1
	Paper	-1 / 1	1 / -1	0 / 0

Pure-strategy and mixed-strategy NE

Player 2			
Player 1		Up	Down
	Up	(2, 2)	(0, 0)
	Down	(0, 0)	(1, 1)

- **Find the 3 Nash Equilibria for this game?**
- **Which equilibrium maximizes the social welfare of the system (sum of payoffs)?**
- Pure strategy NE: action profiles (Down,Down) and (Up,Up) with social welfares of 4 and 2 resp.
- A third Nash equilibrium corresponds to each player choosing action A with probability 1/3 and choosing B with probability 2/3 with a welfare of 2.

Homework

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Online Auctions Outline

- **Introduction to Auctions**
- **Game Theory**
  - Matrix games versus strategic form games
  - I.e., **2**-person games versus **N**-person games
- **Finding Equilibria solutions/outcomes in games**
  - Games with a dominant strategy
  - Pure-strategy Nash Equilibrium (NE)
  - Mixed strategy NE
- **Repeat Games (finite and infinite)**
- **Multi-item auctions (VCG, GSP)**
- **Online Ad Auctions**

Repeat Games:horizon is finite or infinite

- **Goal: maximize payoff if possible**
- **Repeated games may be broadly divided into two classes, depending on whether the horizon is finite or infinite.**
- **May lead to cooperation to improve the welfare of the game**
  - E.g., prisoners dilemma
    - finite leads to selfish ratting-out behaviour (unraveling effect)
    - Whereas as infinite or semi-finite leads to cooperation.

Suspect1\Suspect2	Rat Out	Stay Quiet
Rat Out	-5 / -5	-10 / 0
Stay Quiet	0 / -10	-2 / -2

Nash equilibrium

- **A strategy for each player such that:**
  - No player has an incentive to switch, *if all other players' strategies are held fixed. I.e., will not result in an increase in payoff.*
  - *In our setting each advertiser is a player and each advertiser makes bids (stragegies/moves)*
- **A game could have many Nash equilibria …**
- **E.g., for Rock-Scissors-Paper:**
  - With probability one-third pick each strategy.
- **The Nash Equilibrium can exist both for recurring games and for single-interaction games. If two prisoners are faced with the dilemma once and once alone, their dominant strategy will be to rat each other out.**
- **If two players are in a game like the prisoner's dilemma, but it's played repeatedly, there may be a way for them to cooperate. When a game repeats, the Nash Equilibrium depends on how many times the game is repeated. If it goes on infinitely long term cooperation is easier. If it's only played 3 times, you can imagine how cooperation would be more difficult.**

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Marketplaces

- **The 'real world' in which products and services are provided and consumed**
  - Connecting buyers and sellers from disparate locations
  - E.g., Fish market, flower market, eBay, Craigslist, Google, Yahoo, etc.
- **Markets, or market-like institutions, often allocate goods and services efficiently**
- **Mechanism design theory allows researchers to systematically analyze and compare a broad variety of institutions under various assumptions.**

Mechanisms

- **Leonid Hurwicz (1960) defined a mechanism as a communication system in which participants send messages to each other and/or to a "message center," and where a pre-specified rule assigns an outcome (such as an allocation of goods and services) for every collection of received messages.**
- **Within this framework, markets and market-like institutions could be compared with a vast array of alternative institutions.**

- **These messages may contain private information, such as an individual's (true or pretended) willingness to pay for a public good. The mechanism is like a machine that compiles and processes the received messages, thereby aggregating (true or false) private information provided by many agents.**
- **Each agent strives to maximize his or her expected payoff (utility or profit), and may decide to withhold disadvantageous information or send false information (hoping to pay less for a public good, say).**
- **This leads to the notion of "implementing" outcomes as equilibria of message games, where the mechanism defines the "rules" of the message game. The comparison of alternative mechanisms is then cast as a comparison of the equilibria of the associated message games.**

Incentive efficient

- **To identify an optimal mechanism, for a given goal function (such as profit to a given seller or social welfare), the researcher must first delineate the set of feasible mechanisms, and then specify the equilibrium criterion that will be used**
- **A strategy is dominant if it is a agent's optimal choice, irrespective of what other agents do. to predict the participants' behavior.**
- **the mechanism is <u>incentive-compatible</u> if it is a dominant strategy for each participant to report his private information truthfully.**
- **A direct mechanism is said to be <u>incentive efficient</u> if it maximizes some weighted sum of the agents' expected payoffs subject to their IC**

Mechanism Design

- **Mechanism design is the sub-field of microeconomics and game theory**
- **It considers how to implement good system-wide solutions to problems that involve multiple self-interested agents, each with private information about their preferences.**
  - I.e., advertisers and their bids, auctions, auctioneer, in the case of spot-market online advertising
  - Devise a mechanism for agents to disclose their private information.
- **In recent years mechanism design has found many important applications; e.g., in electronic market design, in distributed scheduling problems, and in combinatorial resource allocation problems.**

Second Price Auction in Online Adv.

- **A publisher is selling a top-right medium rectangle on its homepage (e.g., CNN)**
- **This has some value to potential advertisers**
- **Each Advertiser *k* has his own valuation $v_k \geq 0$ of the ad slot.**
- **The advertisers must SECRETLY simultaneously bid an amount; we denote the bid of buyer i by $b_i \in (0, \infty)$**
- **In a second price auction the highest bidder gets the ad slot and pays the second highest bid.**
  - If there is more than one buyer with the highest bid, the winner is decided by a drawing among the highest bidders and she pays the highest bid.
  - The rest receive a payoff of zero.

2^{nd} Price Auction in Strategic Form Game

- **Given n advertisers, a strategy set for each advertiser is (0, ∞) and a payoff for each advertiser (expected utility function) of the form:**

$$\pi_k = \begin{cases} v_k - s & \text{if } b_k > s \\ 0 & \text{if } b_k < s \\ \frac{1}{r}(v_k - s) & \text{if } k \text{ is among r advertisers with highest bid} \end{cases}$$

$$\textit{where } s \text{ designates the second highest bid (i.e., } s = \max_{i \neq k} b_i)$$

- **Then the strategy profile (v_1,…, v_n) is a Nash Equilibrium for this game (i.e., Truth telling is a Nash Equilibrium!!)**

[Games and Decision Making, by Charalambos D. Aliprantis, Subir Kumar Chakrabarti]

Truth Telling is the Nash Equilibrium of SPA

- **Proof: Two Scenarios (always want positive payoff)**
  - Assume advertiser bid b_i for an ad slot that is of value v_i to the advertiser)
  - **Scenario 1**: An advertiser *i* never gains by **bidding more** than the true value for that advertiser [An advertiser i never gains by bidding $b_i > v_i$; And assume $b_i > v_i$ and let $b_{-i} = \max_{j \neq i} bj$]
  - **Scenario 2**: An advertiser *i* never gains by bidding less than the true value for that advertiser [An advertiser i never gains by bidding $b_i < v_i$]

Scenario 1: advertiser *i* never gains by bidding more than the true value (i.e., $b_i > v_i$)

- **CASE 1: $b_{-i} > b_i$**
  - Some other bidder has the highest bid and so player *i* gets zero, which he could get by bidding v_i.
- **CASE 2: $v_i < b_{-i} < b_i$ [gets payoff < 0]**
  - Bidder *i* wins and gets $v_i - b_{-i} < 0$. However, if he would have bid v_i, then his payoff would have been zero—a higher payoff than that received by bidding bi.
- **CASE 3: $b_{-i} = b_i$**
  - Here bidder *i* is one among r buyers with the highest bid and he receives $(v_i - b_{-i})/r < 0$. But, by bidding v_i he can get 0, a higher payoff.
- **CASE 4: $b_{-i} < v_i$**
  - In this case bidder *i* gets $v_i - b_{-i}$ which he could get by bidding v_i.
- **CASE 5: $b_{-i} = v_i$**
  - Here again bidder *i* is one among r buyers with the highest bid and he receives $v_i - b_{-i}\ r = 0$. But, by bidding v_i he can also get 0.

Truth Telling is the Nash Equilibrium of SPA

- **Proof: Two Scenarios (always want positive payoff)**
  - Assume advertiser bid b_i for an ad slot that is of value v_i to the advertiser)
  - **Scenario 1**: An advertiser *i* never gains by **bidding more** than the true value for that advertiser [An advertiser i never gains by bidding $b_i > v_i$; And assume $b_i > v_i$ and let $b_{-i} = \max_{j \neq i} bj$]
  - **Scenario 2**: An advertiser *i* never gains by bidding less than the true value for that advertiser [An advertiser i never gains by bidding $b_i < v_i$]

Scenario 2: advertiser *i* never gains by bidding LESS than the true value (i.e., $b_i < v_i$)

- **If $b_{-i} > v_i$ then bidder $_i$ would have a zero payoff which is the same as the payoff she would get if she bid v_i.**
- **On the other hand, if $b_{-i} \leq vi$, then player *i* would do at least as well if she bid v_i.**

- **In summary, the strategy profile $(v_1, v_2, . . . , v_n)$ is a Nash equilibrium.**
  – Therefore, it is reasonable to expect that every advertiser will bid their true valuation of the ad slot and the advertiser with the highest valuation wins. Note that this is true even if the advertisers do not know the valuation of the other bidders.
- **Truth telling is a Nash Equilibrium of 2nd Price Auction !!**
  – Helps Advertisers avoid time-consuming strategic game playing and ensures that the ad slot is sold to the advertiser that values it the most

First Price Optimal Bidding Strategy

- **First-Price Sealed**
  - Need to ***shade*** your bid in order to make a profit.
    - ***Shading***: is when you place a bid less than your value, *V* but not so low as to guarantee losing. Involves risk for reward.
  - I.e., not truth telling

$$\pi_k = \begin{cases} v_k - b_k & \text{if } b_k > s \\ 0 & \text{if } v_k < s \end{cases}$$

$$\textit{where } s \text{ designates the second highest bid (i.e., } s = \max_{i \neq k} b_i)$$

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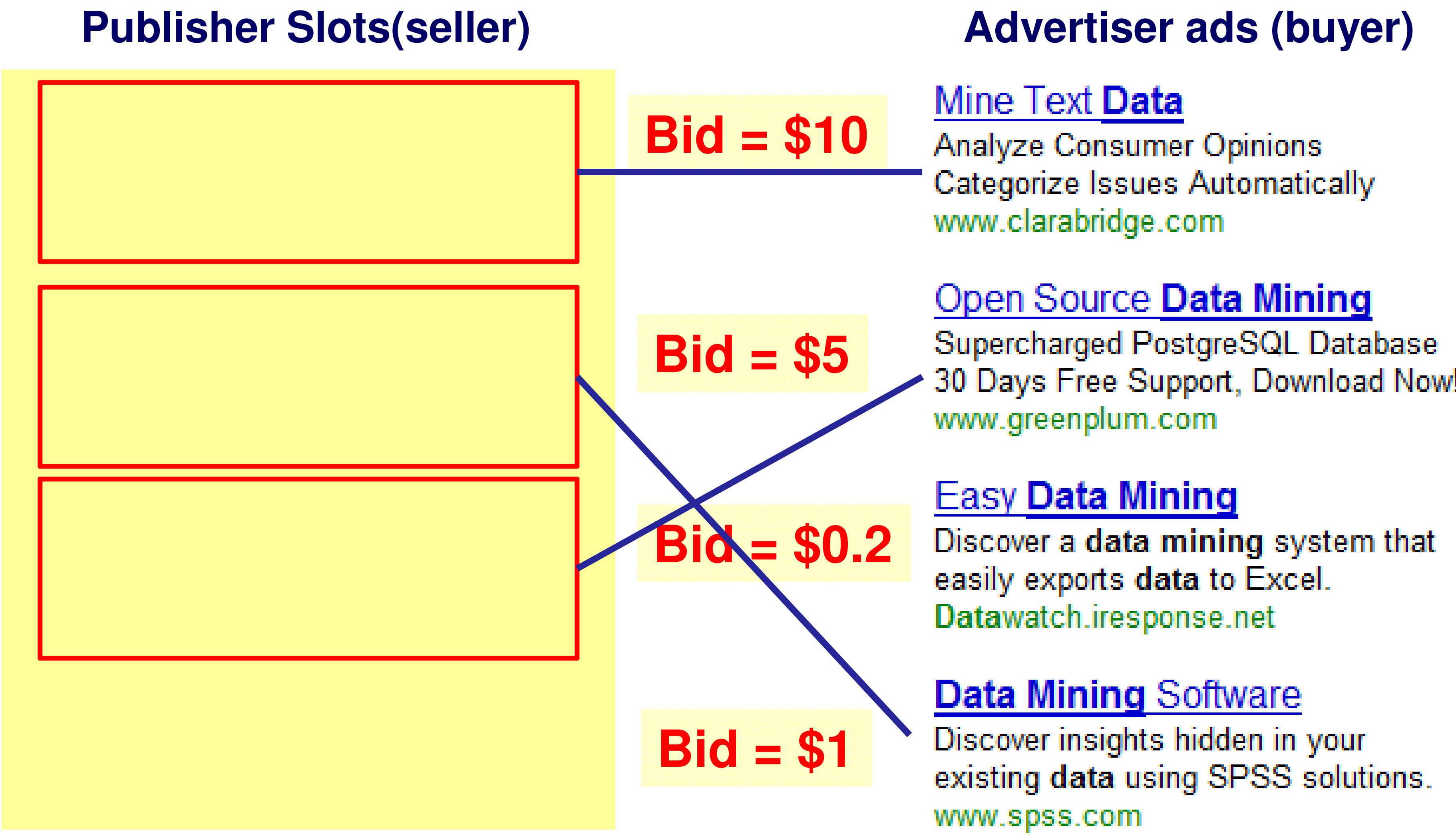
Generalized Auction (Multi-item)



View as an bipartite graph; encode as a network.

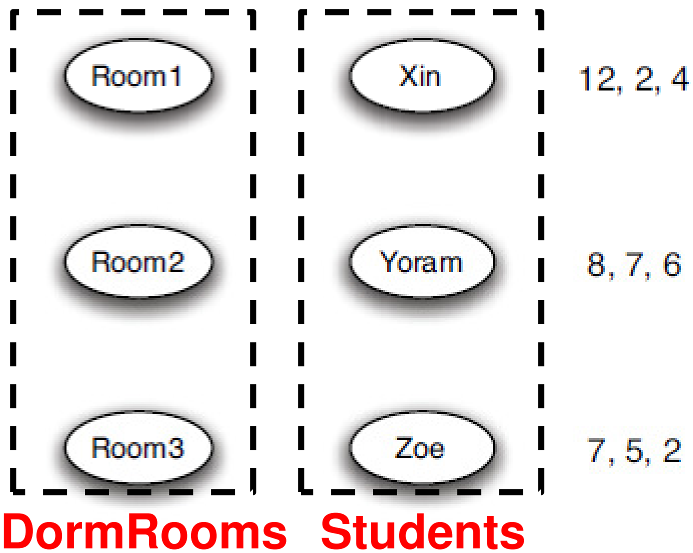
Multi-item Auction: Bipartite Matching

- **A bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V***
  - such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets.
- **Perfect Matching:**
  - When there are an equal number of nodes on each side of a bipartite graph, a perfect matching is an assignment of nodes on the left to nodes on the right, in such a way that
  - (i) each node is connected by an edge to the node it is assigned to, and
  - (ii) no two nodes on the left are assigned to the same node on the right.
- **Market Clearing**
  - A set of assignments (sell, buy) such that each buyer that maximizes their payoff and only one item goes to each buyer

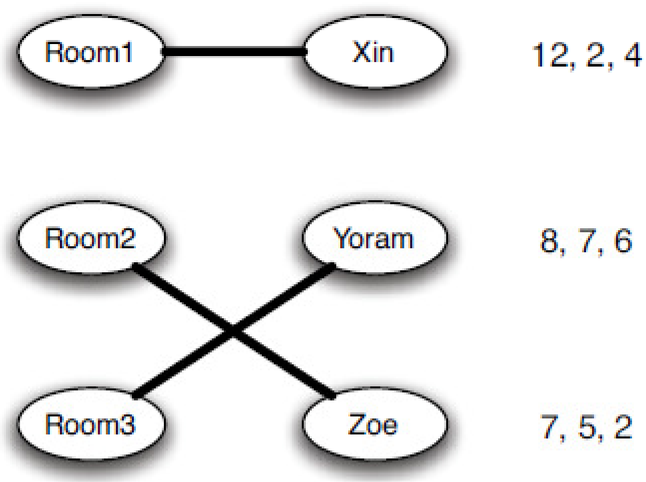
[Easley and Kleinberg, 2010]

Optimal Assignment

- **Optimal assignment: maximizes the total happiness/valuation of everyone (though it does not give everyone their favorite item).**
- **Administrator performs the assignment (12+6+5=23)**



(a) *A set of valuations*

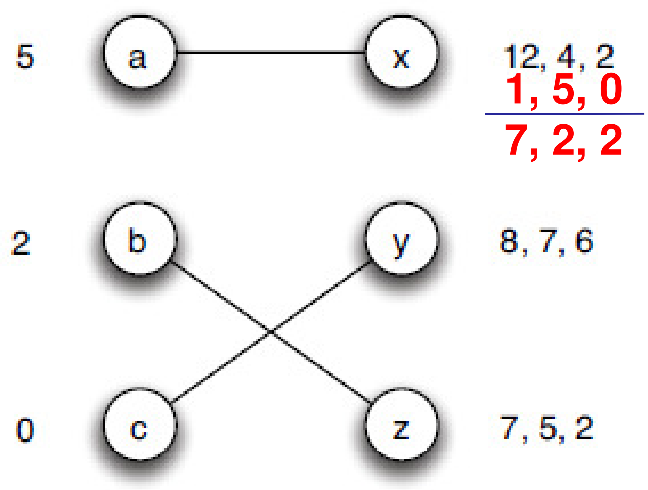


(b) *An optimal assignment*

Figure 10.3: (a) A set of valuations: each person's valuations for the objects appears as a list next to them. (b) An optimal assignment with respect to these valuations.

[Easley and Kleinberg, 2010]

Multi-item Auctions

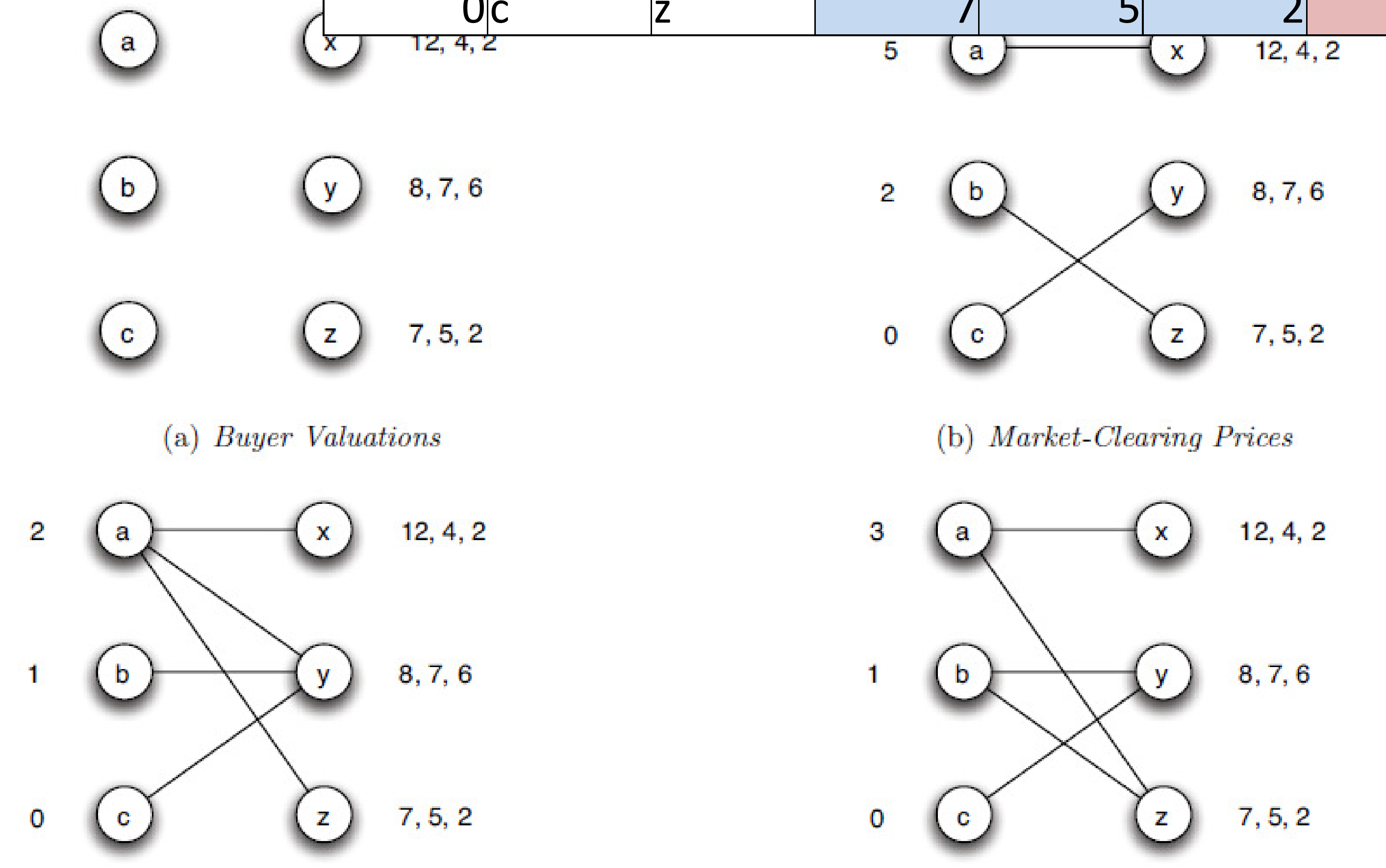


Seller announces prices (5, 2, 0)
Payoff = v_{ij} - p_i
Preferred seller is the seller that maximizes the payoff for the buyer
A set of prices is market clearing if the resulting preferred-seller graph has a perfect match (1 to 1 assignment).

Price(pi)	Seller(i)	Buyer(j)	Valuations(vij)			Payoffs_ij=vij-pi		
5	a	x	12	4	2	7	2	2
2	b	y	8	7	6	3	5	6
0	c	z	7	5	2	2	3	2

[Easley and Kleinberg, 2010]

Price(pi)	Seller(i)	Buyer(j)	Valuations(vij)			Payoffs_ij=vij-pi		
5	a	x	12	4	2	7	2	2
2	b	y	8	7	6	3	5	6
0	c	z	7	5	2	2	3	2



(a) *Buyer Valuations*

(b) *Market-Clearing Prices*

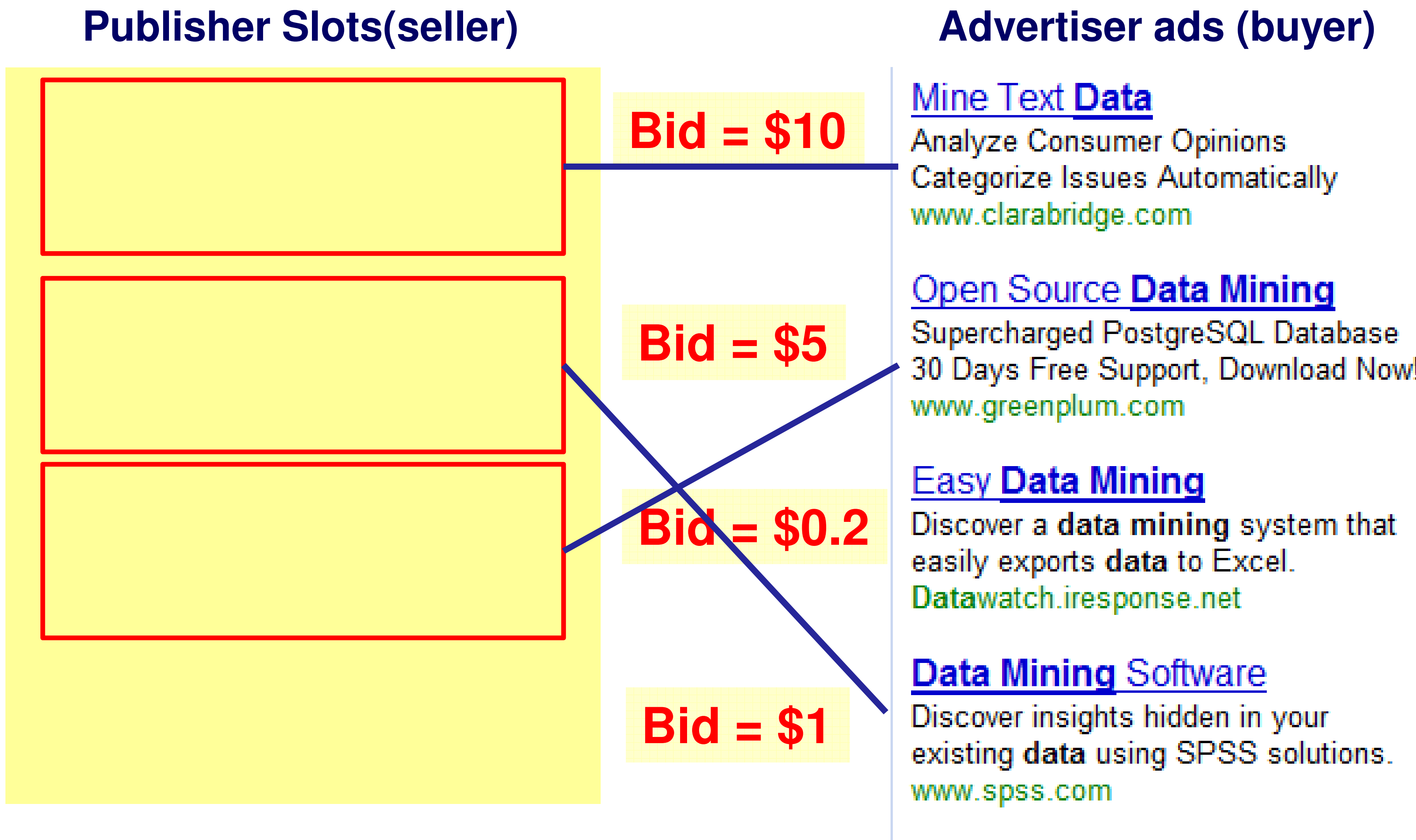
(c) *Prices that Don't Clear the Market*

(d) *Market-Clearing Prices (Tie-Breaking Required)*

[Easley and Kleinberg, 2010]

Figure 10.5: (a) Three sellers (a, b, and c) and three buyers (x, y, and z). For each buyer node, the valuations for the houses of the respective sellers appear in a list next to the node. (b) Each buyer creates a link to her preferred seller. The resulting set of edges is the preferred-seller graph for this set of prices. (c) The preferred-seller graph for prices 2, 1, 0. (d) The preferred-seller graph for prices 3, 1, 0.

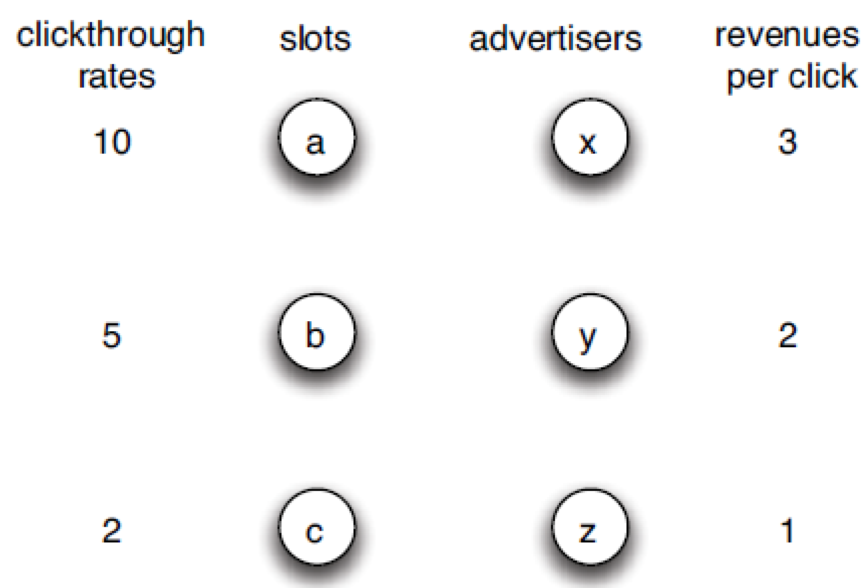
Generalized Auction (Multi-item)



View as an bipartite graph; encode as a network.

Assign slots to ads using matching markets

- **Assume**
  - Advertisers know the CTRs
  - CTR depends on slot only (and not the ad that is shown there)
  - CTR of a slot does not depend on the ads that in shown in the other slots (complex to analyze)



VCG Principle - SPA

- **The second-price auction produces an allocation that maximizes social welfare — the bidder who values the item the most gets it.**
- **The winner of the auction is charged an amount equal to the "harm" (missed opportunity) he causes the other bidders by receiving the item.**
- **Suppose the bidders' values for the item are *v1, v2, v3, . . . , vn* in decreasing order. Then if bidder *1* were not present, the item would have gone to bidder *2*, who values it at *v2*.**
  - Other bidders still would not get the item, even if bidder 1 weren't there.
  - Thus bidders 2 through n collectively experience a harm of v2 (or a missed opportunity of v2)
  - This harm of v2 is what bidder 1 is charged a second price auction
  - Other bidders are also charged the harm they cause (i.e., zero in this single-item auction)

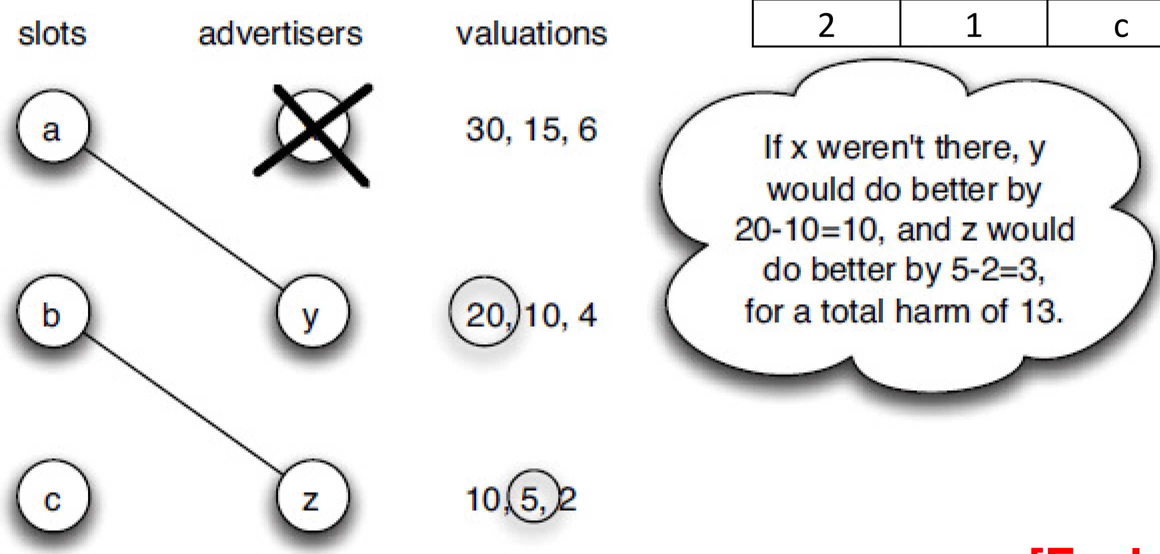
VCG Principle and Multi-item Auctions

- **In a matching market, we have a set of buyers and a set of sellers and buyer *j* has a valuation of v_{ij} for the item being sold by seller *i*. Buyers have independent, private values.**
- **1: assign items to buyers so as to maximize total valuation.**
- **2: the price buyer *j* should pay for seller *i*'s item — in the event she receives it — is the harm she causes to the remaining buyers through her acquisition of this item.**
  - This is equal to the total boost in valuation everyone else would get if we computed the optimal matching without buyer j present.
  - (A matching that maximizes the total payoff is also one that maximizes the total valuation pg 237 EK)

How much should advertiser X pay?

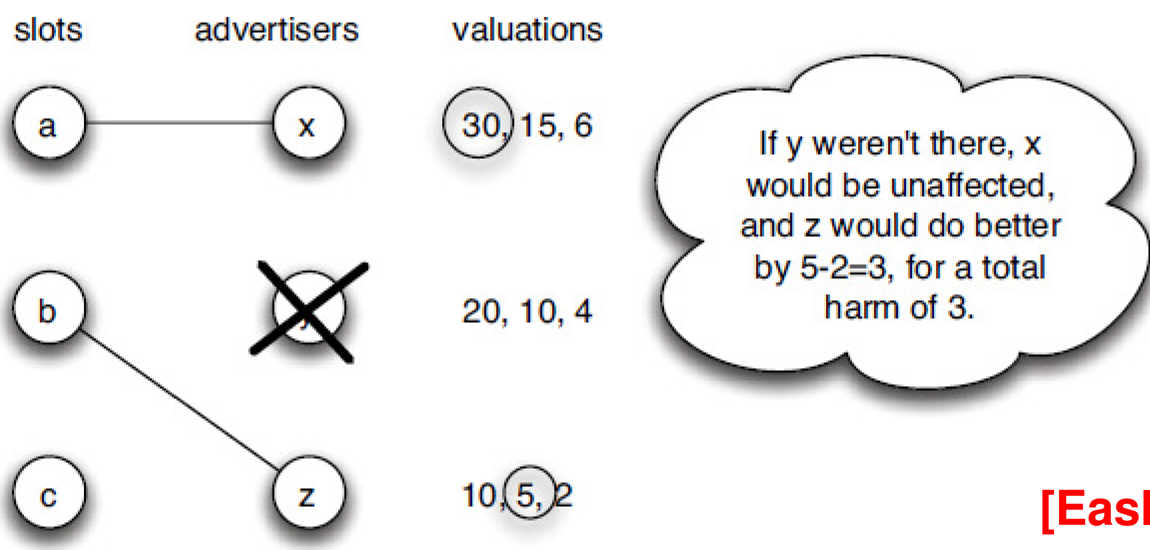
- In an optimal matching without x present, advertisers y and z gets slot a and b respectively. This improves the respective valuations of y and z for their assigned slots (by 10 and 3).
- So x should pay the harm that she causes to y and z (i.e., 13=10 +3)

#Clicks	Rev per click	Seller(i)	Buyer(j)	Valuations(vij)		
10	3	a	x	30	15	6
5	2	b	y	20	10	4
2	1	c	z	10	5	2



[Easley and Kleinberg, 2010]

How much should advertiser y pay?



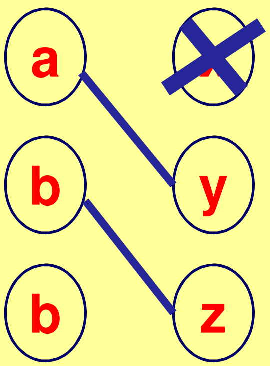
[Easley and Kleinberg, 2010]

VCG Prices for general market

- **VCG price p_{ij} that ad(buyer) *j* pays for item *i*.**
  - (view sellers and buyers as slots and ads)
  - Let *M* denote the set of slots and *N* the set of ads
  - Let V_N^M denote the maximum total valuation over all possible perfect matchings of slots and ads (1 ad slot for each ad; note some ad slots will be null ad slots)
    - this is simply the value of the socially optimal outcome with all slots and ads present
  - Let *M-i* denote the set of slots with i removed and let *N-j* denote the set of ads with ad j removed
  - If we sell slot *i* to ad *j* then the total best valuation of the rest of the ads could get is V_{N-j}^{M-i}
  - If ad *j* did not exist but slot *i* were still an option for all other ads then the best valuation is: $p_{ij} = V_{N-j}^{M} - V_{N-j}^{M-i}$

VCG Payment of P_{11}

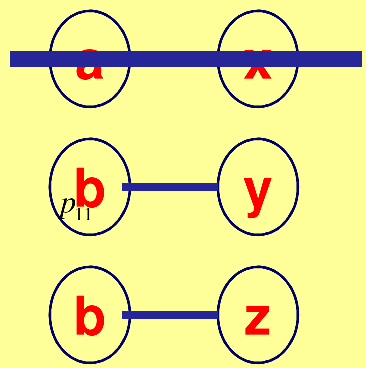
Optimal matching of slots and ads and valuation with ad *x* removed



Valuations(vij)		
30	15	6
20	10	4
10	5	2

$$V_{N-j}^{M} = 25$$

Optimal matching of slots and ads and valuation with slot *a* and ad *x* removed



Valuations(vij)		
30	15	6
20	10	4
10	5	2

$$V_{N-j}^{M-i} = 12$$

$$p_{ij} = V_{N-j}^{M} - V_{N-j}^{M-i}$$

$$p_{11} = V_{N-1}^{M} - V_{N-1}^{M-1}$$

$$p_{11} = 25 - 12 = 13$$

What is the VCG payment for P_{22} ? (0+3)

VCG Auction Summary

1. **Advertisers submit their sealed bids**
2. **Auctioneer chooses a social optimal assignment of slots to ads**
   - I.e., a perfect matching that maximizes the total valuation of each buyer for what they get. This assignment is based on the announced valuations (since that's all they have access to.)
3. **Charge each advertiser the appropriate VCG price** $p_{ij} = V^{M}_{N-j} - V^{M-i}_{N-j}$

This auction is a game that the advertisers play: they must choose a strategy (a set of valuations to announce), and they receive a payoff: their valuation for the slot they get, minus the price they pay.
What turns out to be true, though it is far from obvious, is that this game has been designed to make <u>truth-telling</u> — in which a buyer announces her true valuations — a dominant strategy.

Dominant Strategy for VCG Auction

- …is truthtelling
- If items are assigned and prices computed according to the VCG procedure, then truthfully announcing valuations is a dominant strategy for each buyer, and the resulting assignment maximizes the total valuation of any perfect matching of slots and advertisers.

- See Easley and Kleinberg 2010 for details

Generalized Second Price (GSP) Auction

- GSP — like VCG — is a generalization of the second-price auction for a single item.
- However, as will see, GSP is a generalization only in a superficial sense, since it doesn't retain the nice properties of the second-price auction and VCG (i.e., truthtelling)
- Introduced by Google in 2002.

GSP Auction Summary

1. **Advertisers submit their sealed bids**
2. **Auctioneer awards each slot *i* to the *i*th highest bidder,**
3. **And charges a price per click equal to the (i + 1)st highest bid.**
   - In other words, each advertiser who is shown on the results page is paying a price per click equal to the bid of the advertiser just below them.

$$p_{ij} = v_{i+1} + \$0.01$$

This auction is a game that the advertisers play: they must choose a strategy (a set of valuations to announce), and they receive a payoff: their valuation for the slot they get, minus the price they pay.

This GSP game <u>does not</u> have a dominant strategy in <u>truth-telling</u>.

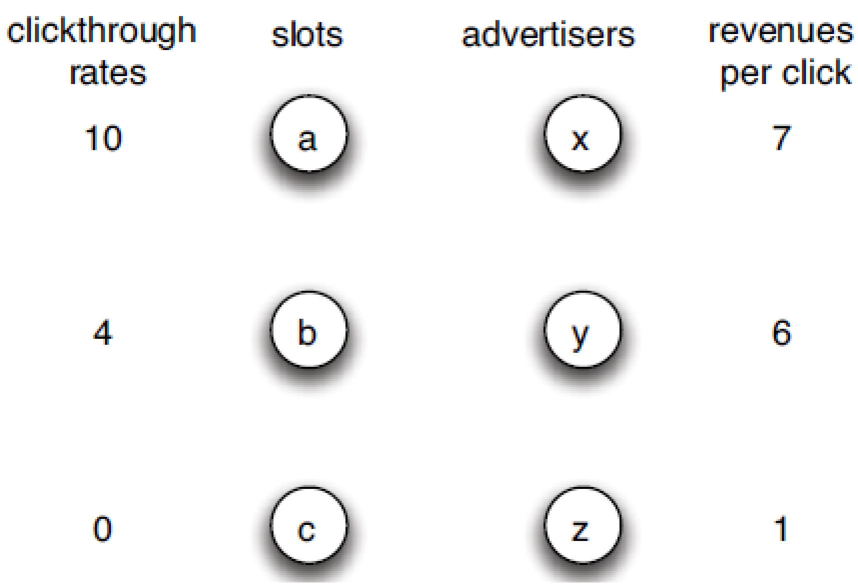
Analyzing GSP:

- **Formulate GSP as a game,**
  - Each advertiser is a player, its bid is its strategy, and its payoff is its revenue minus the price it pays.
- **GSP may have multiple and non-optimal equilibra**
- **Truth-telling may not be an equilibrium (see next slide)**
- **In this game, we will consider Nash equilibria**
  - we seek sets of bids so that, given these bids, no advertiser has an incentive to change how it is behaving
  - In order to analyze Nash equilibrium in the bidding game we will assume that each advertiser knows the values of all other bidders. Otherwise, they do not know the payoffs to all players in the bidding game and we could not use Nash equilibrium to analyze the game. The motivation for this assumption is that we envision a situation in which these bidders have been bidding against each other repeatedly and have learned each others' willingnesses to pay for clicks.

[Varian 2006] and [Edelman, Ostrovsky,and Schwarz 2005]

Truth-telling may not be an equilibrium

- **If each advertiser bids its true valuation then x gets the top slot at a PPC of $6 (x pays a cumulative price of $60. Yield a payoff for x is 10*$7 – 10*$6 = $10**
- **Now if x lowers its bid to $5 thereby implying a cumulative price of $4 for the slot.**
  - And a payoff of $7*4 - $1*4=$24
- **This is an improvement over bidding truthfully (and therefore incentive to lower bid, (shade or lie))**



[Easley and Kleinberg, 2010]

Dominant strategy of GSP is not truth-telling

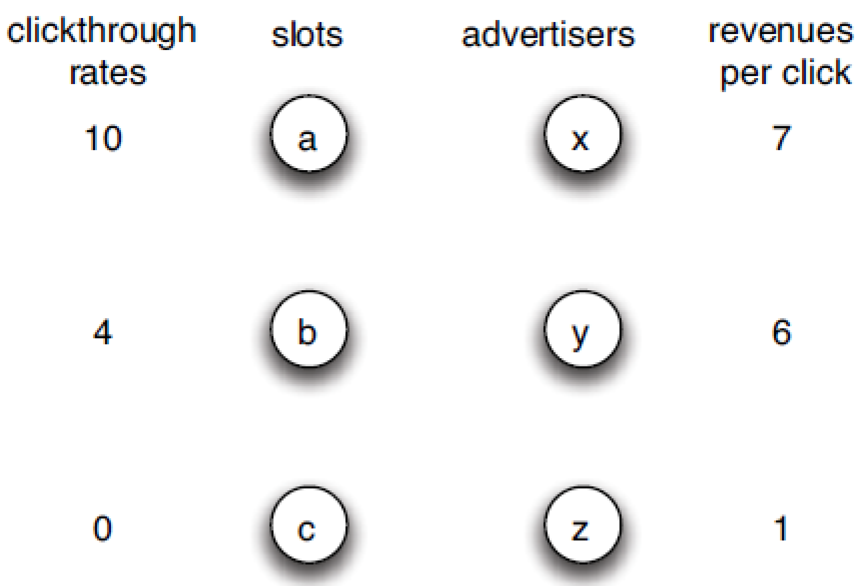
Another example

- **Assume three bidders, with values per click of \$10, \$4, and \$2, and two positions. However, the clickthrough rates of these positions are now almost the same: the first position receives 200 clicks per hour, and the second one gets 199.**
- **If all players bid truthfully, then bidder 1's payoff is equal to (\$10 – \$4) ∗ 200 = \$1200.**
- **If, instead, bidder 1 shades his bid and bids only \$3 per click, he will get the second position, and his payoff will be equal to (\$10 – \$2) 199 = \$1592 > \$1200.**
- **(so bidder1 is very incentivized to change strategy/bid)**

[Edelman, Ostrovsky,and Schwarz 2005]

GSP can have multiple Equilibria

- **$5, $4, $1 forms a Nash equilibrium (trust but verify) (socially optimal)**
  - x doesn't want to lower its bid to $4 so as to move to the second slot, and y doesn't want to raise its bid to $5 to get the first slot.
- **So does $3, $5, $1 (again trust but verify) (not socially optimal)**
- **The existence of multiple equilibria also adds to the difficulty in reasoning about the search engine revenue generated by GSP, since it depends on which equilibrium (potentially from among many) is selected by the bidders.**



Bid	Payoff	Bid	Payoff
5	?	3	?
4	?	5	?
1	?	1	?

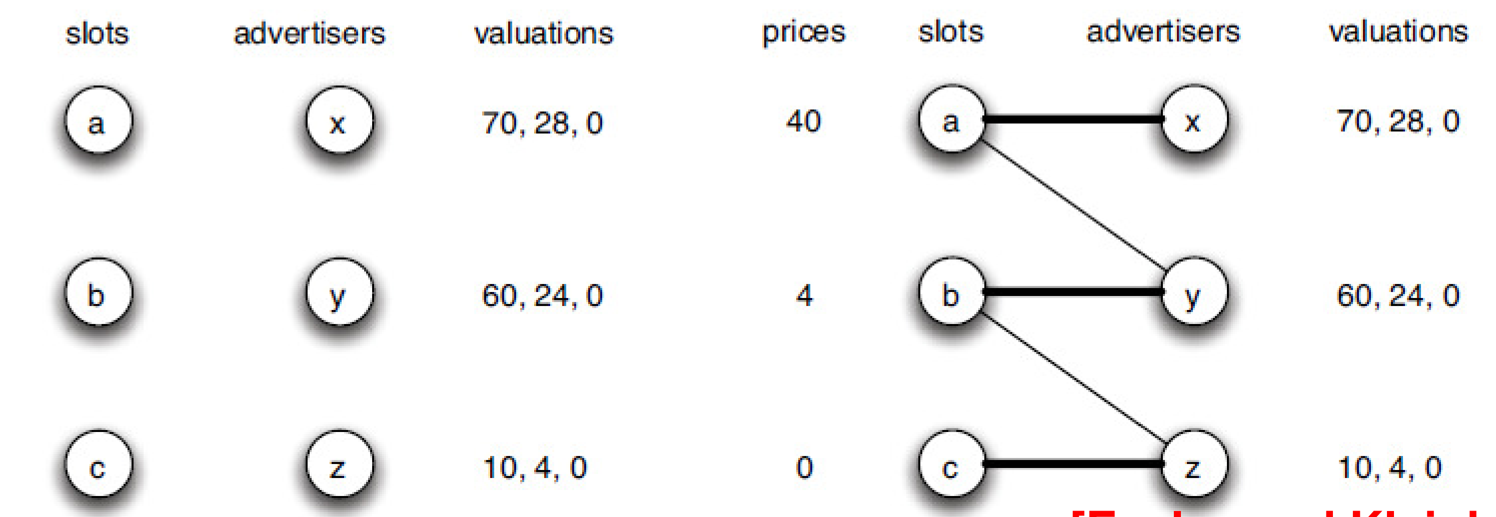
[Easley and Kleinberg, 2010]

Socially Optimal Assignment

- **Socially optimal assignment of slots to ads — that is, a perfect matching that maximizes the total valuation (and at the same time maximizes the payoffs for each buyer for what they get)**
- **This assignment is based on the announced advertiser valuations.**

Finding a socially optimal equilibrium

- **The socially optimal one can be easily constructed by following a few simple principles, rather than by trial-and-error or guesswork**
- **1: we get a set of advertiser valuations for slots**
- **2: we produce a set of market-clearing prices**
- **3: These are cumulative prices for each slot—single prices that cover all the clicks associated with that slot.**
  - We can easily translate back to prices per click by simply dividing by the clickthrough rate: this produces a price per click of $40/10 = $4 for the first slot and $4/4=$1 for slot 2; $0 for slot 3



[Easley and Kleinberg, 2010]

(a) *Advertiser valuations for the previous ex-* (b) *Market-clearing prices for the previous example.*

T_com

Strategy/Bid selection

- **Next, we find bids that result in these prices per click. This is not hard to do: the prices per click are $4 and $1 for the two slots, so these should be the bids of *y* and *z* respectively.**
- **Then the bid of x can be anything as long as it's more than 4.**
- **With these bids, x pays $4 per click for the first slot, y pays $1 per click for the second slot, and z pays $0 per click for the (fake) third slot — and the allocation of advertisers to slots is socially optimal.**
- **Show that bids form a Nash Eqilibrium**
  - Advertisers have no incentive to increase or decrease their bids

GSP and Locally Envy-Free Equilibria

- **How do advertiser's reach an equilibrium?**
- **Advertisers originally have private information; gradually learn the values of others and can adjust their bids frequently**
  - Always give it your best shot otherwise why bid
  - Bid vectors converge to an equilibrium such that neighboring bidders have no incentive to change (with VCG payoffs)

Definition 4 *An equilibrium of the static game induced by GSP is* locally envy-free *if a player cannot improve his payoff by exchanging bids with the player ranked one position above him. More formally, in a locally envy-free equilibrium, for any* $i \leq \min\{N, K\}$, *we have* $\alpha_i s_{g(i)} - p^{(i)} \geq \alpha_{i-1} s_{g(i)} - p^{(i-1)}$.

[Edelman, Ostrovsky,and Schwarz 2005]

GSP Revenue ≥ VCG Revenue

- **Search engines and ad networks are motivated to choosing a procedure that will maximize their revenue (given the behavior of advertisers)**
- **Search engines and ad networks may not wish to know the true value of a click**
- **So GSP works well in practice !**

Theorem 7 *Strategy profile B^* is a locally envy-free equilibrium of game Γ. In this equilibrium, each bidder's position and payment is equal to those in the dominant-strategy equilibrium of the game induced by VCG. In any other locally envy-free equilibrium of game Γ, the total revenue of the seller is at least as high as in B^*.*

Dominant strategy of GSP is not truth-telling

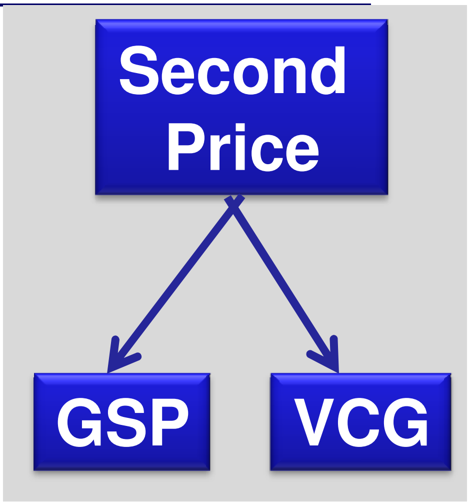
- **"GSP does not have an equilibrium in dominant strategies, and truth-telling is generally not an equilibrium strategy" [Edelman, Ostrovsky, Schwarz, 2006]**
- **Static equilibrium of GSP is locally envy-free: no advertiser can improve his payoff by exchanging bids with advertiser in slot above**
- **GSP is Complex**
  - $\text{Revenue}_{VCG} \leq \text{Revenue}_{GSP}$
    - Advertiser will generally be over paying (since truth-telling is an equilibrium of VCG)
    - Truth-telling is not an equilibrium of GSP (so Search engine will not know true value; maybe a good thing)

Truth Telling: Standard Second Price

- **Truth-telling is a dominant strategy for standard second price (AKA Vickrey) auction**
  - Single ad slot for sale; highest bidder *i* pays the bid of the second highest bidder (bid_{i+1})
- **Generalised second price GSP Auction (multiple slots)**
  - The dominant strategy of GSP (multiple slots) is not truth-telling
  - Bidder i pays the bid price of the next ranked bid bid_{i+1}

- **Vickrey-Clarke-Groves (VCG) mechanism**
  - Another generalization of the Vickrey auction that maintains the incentive to bid truthfully
  - The idea in VCG is that **each player in the auction pays the opportunity cost that their presence introduces to <u>all</u> the other players**.

First price (GFP) vs. Second Price GSP



- **Generalized First Price Auction**
  - Unstable
- **Second Price Auction (Single Item)**
  - Truth-telling is the dominant strategy
  - (i.e., no buyer's remorse when bidding true value)
- **Generalized 2nd Price (GSP) Auction**
  - Tailored to the unique environment of online ads [Google, 2002]
  - BUT truth-telling is NOT a dominant strategy for Generalized Second Price (GSP) Auctions [Edelman et al. 2006]
- **Vickrey, Clarke, Groves (VCG) Auction**
  - Truth-telling is a dominant strategy under VCG
  - In particular, unlike the VCG mechanism, GSP generally does not have an equilibrium in dominant strategies and truth-telling is not an equilibrium of GSP.

Online Auctions Outline

- **Introduction to Auctions**
- **Game Theory**
  - Matrix games versus strategic form games
  - I.e., **2**-person games versus **N**-person games
- **Finding Equilibria solutions/outcomes in games**
  - Games with a dominant strategy
  - Pure-strategy Nash Equilibrium (NE)
  - Mixed strategy NE
- **Repeat Games (finite and infinite)**
- **Multi-item auctions (VCG, GSP)**
- **Online Ad Auctions**

Keyword Auction Systems: Goto Model

- **Rank ads by keyword bid price**
  - each ad is associated with multiple keywords; assume one keyword for now and exact match
- **In 1997, Goto/Overture (now Yahoo! Search Marketing) launched an innovative framework for selling advertising space next to search results.**
  - Rather than selling large, expensive chunks of advertising space (human sales force), each keyword was sold via its own auction
  - Human editors checked for relevance
  - Payment was made on a pay-per-click (PPC)
  - Used a **first price auction mechanism** (and published the winning bids!!)
  - Successful; advertising system adapted by Yahoo and MSN

Generalized first-price auction (GFP)

- **For each keyword, several advertising slots are auctioned at once, each one representing a position relative to the top of the search page.**
- **Overture created a marketplace around each keyword**
  - Their auction mechanism has been characterized as a generalized first-price auction (GFP) .
  - Each advertiser submits a ***secret bid*** (value of click/action) to the auctioneer (Overture in this case).

1st Price – In a first-price auction for a ***single item***, the highest bidder wins the item at the highest price.

GFP – In a **GFP**, ***multiple items*** are up for auction; the highest bidder wins the first item at the highest price, the second-highest bidder wins the second item at the second-highest price, and so on.

Generalized First Price Auction

1. In a GFP, multiple items are up for auction;
2. The highest bidder wins the first item at the highest price
3. The second-highest bidder wins the second item at the second-highest price, and so on

KW Bid = $10

Mine Text **Data**
Analyze Consumer Opinions
Categorize Issues Automatically
www.clarabridge.com

KW Bid = $5

Open Source **Data Mining**
Supercharged PostgreSQL Database
30 Days Free Support, Download Now!
www.greenplum.com

KW Bid = $2

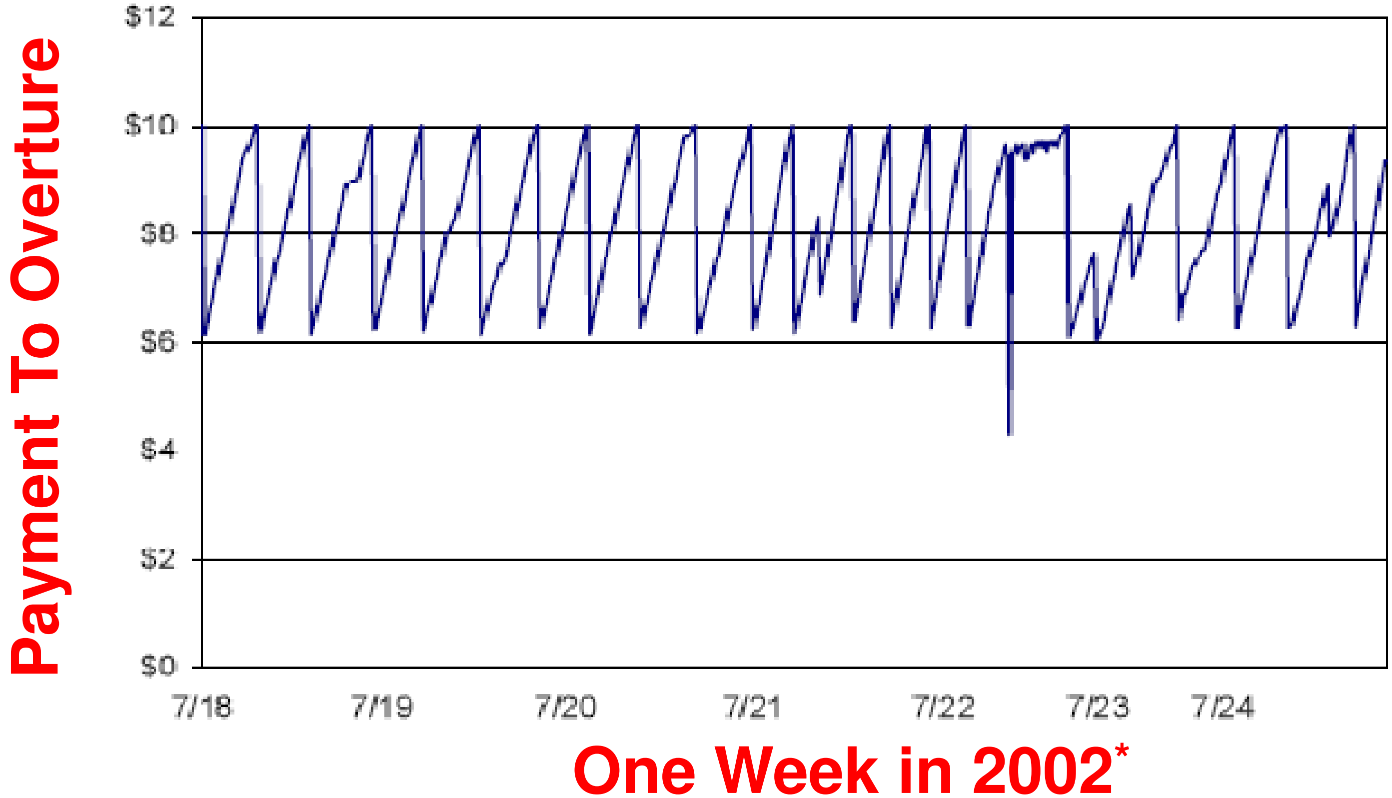
Easy **Data Mining**
Discover a **data mining** system that easily exports **data** to Excel.
Datawatch.iresponse.net

KW Bid = $1

Data Mining Software
Discover insights hidden in your existing **data** using SPSS solutions.
www.spss.com

Gaming the system: GFP not stable

- **Another notable aspect of Overture's auction design was that winning bids were posted**
- **Led to buyer's remorse and gaming systems; no equilibrium**



*[Edelman, B. et al.Internet advertising and the generalized second price auction: selling billions of dollars worth of keywords. NBER Paper No. W11765, 2005]

Generalized 2nd Price (GSP) Auction

1. In a GSP, multiple items are up for auction;
2. The highest bidder wins the first item at the second price (+delta)
3. The second-highest bidder wins the second item at the third-highest price, and so on

Bid = $10
PPC = $5

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Analyze Consumer Opinions
Categorize Issues Automatically
www.clarabridge.com

Bid = $5
PPC = $2

Open Source **Data Mining**
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www.greenplum.com

Bid = $2
PPC = $1

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Datawatch.iresponse.net

Bid = $1
PPC = $0.57

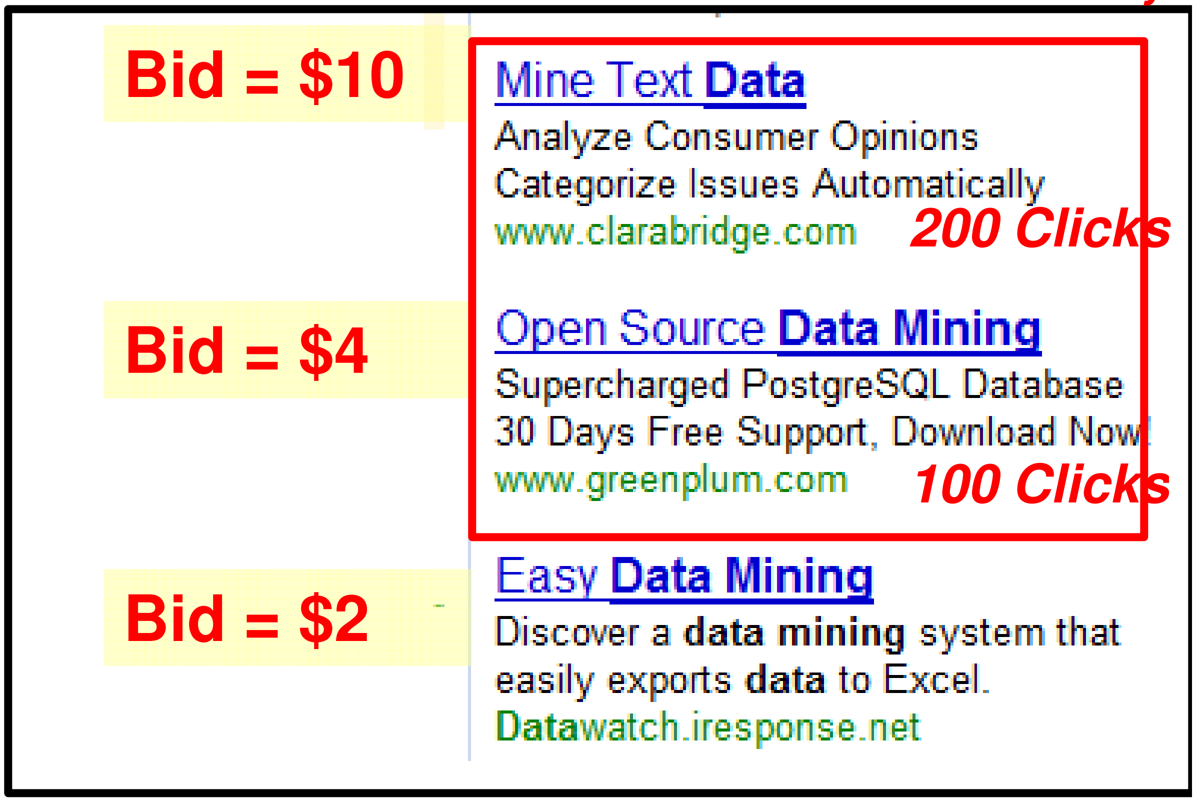
Data Mining Software
Discover insights hidden in your existing **data** using SPSS solutions.
www.spss.com

Introduced by Google in Feb 2002 (AdWords); overcomes the instability of GFP because by design the bidder is incentivized to pay the true value?!

Example Auction

Note:
However, in a GSP/VCG auction, advertisers must submit a single bid even though there are several advertisement slots available.

Assume 2 ads slots only



Suppose there are two slots on a page and three advertisers. An ad in the first slot receives 200 clicks per hour, while the second slot gets 100.

Generalized 2nd Price (GSP) Auction

Assume 2 ads slots only

1. In a GSP, multiple items are up for auction;
2. The highest bidder wins the first item at the second price (+delta)
3. The second-highest bidder wins the second item at the third-highest price, and so on

Bid = $10
PPC = $4
Payment =$4*200

Mine Text **Data**
Analyze Consumer Opinions
Categorize Issues Automatically
www.clarabridge.com ***200 Clicks***

Bid = $4
PPC = $2
Payment =$2*100

Open Source **Data Mining**
Supercharged PostgreSQL Database
30 Days Free Support, Download Now!
www.greenplum.com ***100 Clicks***

Bid = $2
PPC = $2

Easy **Data Mining**
Discover a **data mining** system that easily exports **data** to Excel.
Datawatch.iresponse.net

Revenues under GSP is $1,000

VCG Auction: Externality Cost

Assume 2 ads slots only

Bid = $10
PPC = $3
Payment =$600

Bid = $4
PPC = $2
Payment =$200

Bid = $2

VCG Rev is $800
VCG Rev ≤ GSP ($1,000)

$$PaymentVCG_i = (Clicks_i - Clicks_{i+1}) * Bid_{i+1} + PaymentVCG_{i+1}$$

1. **Where revenue for slot 1 is $600.**
   - **$200 for the externality that he imposes on advertiser 3 (by forcing him out of position 2) and**
   - **$400 for the externality that he imposes on advertiser 2 (by moving him from position 1 to position 2 and thus causing him to lose (200–100) = 100 clicks per hour).**
2. **Revenue for slot 2 is $200 (same in VCG and GSP)**

VCG Auction: Externality Cost

Assume 2 ads slots only

Bid = $10
PPC = $3
Payment =$600

[Mine Text **Data**]
Analyze Consumer Opinions
Categorize Issues Automatically
www.clarabridge.com *200 Clicks*

Bid = $4
PPC = $2
Payment =$200

[Open Source **Data Mining**]
Supercharged PostgreSQL Database
30 Days Free Support, Download Now!
www.greenplum.com *100 Clicks*

Bid = $2

[Easy **Data Mining**]
Discover a **data mining** system that

VCG Rev is $800
VCG Rev ≤ GSP ($1,000)

$$PaymentVCG_i = \left(Clicks_i - Clicks_{i+1}\right) * Bid_{i+1} + PaymentVCG_{i+1}$$

1. **Where revenue for slot 1 is $600.**
   - **$200 for the externality that he imposes on advertiser 3 (by forcing him out of position 2) and**
   - **$400 for the externality that he imposes on advertiser 2 (by moving him from position 1 to position 2 and thus causing him to lose (200–100) = 100 clicks per hour).**
2. **Revenue for slot 2 is $200 (same in VCG and GSP)**

$$PaymentVCG_i = (Clicks_i - Clicks_{i+1}) * Bid_{i+1} + PaymentVCG_{i+1}$$

Rank	#clicks	Bid price	Externality Costs for ad B	Externality Costs for ad C
1 (A)	200	10	(200-100)*4=$400	
2 (B)	100	4	100*2=$200	100*2 ($\#Click_N * b_{N+1}$)
3 (C)	0	2	$600 total	$200 total

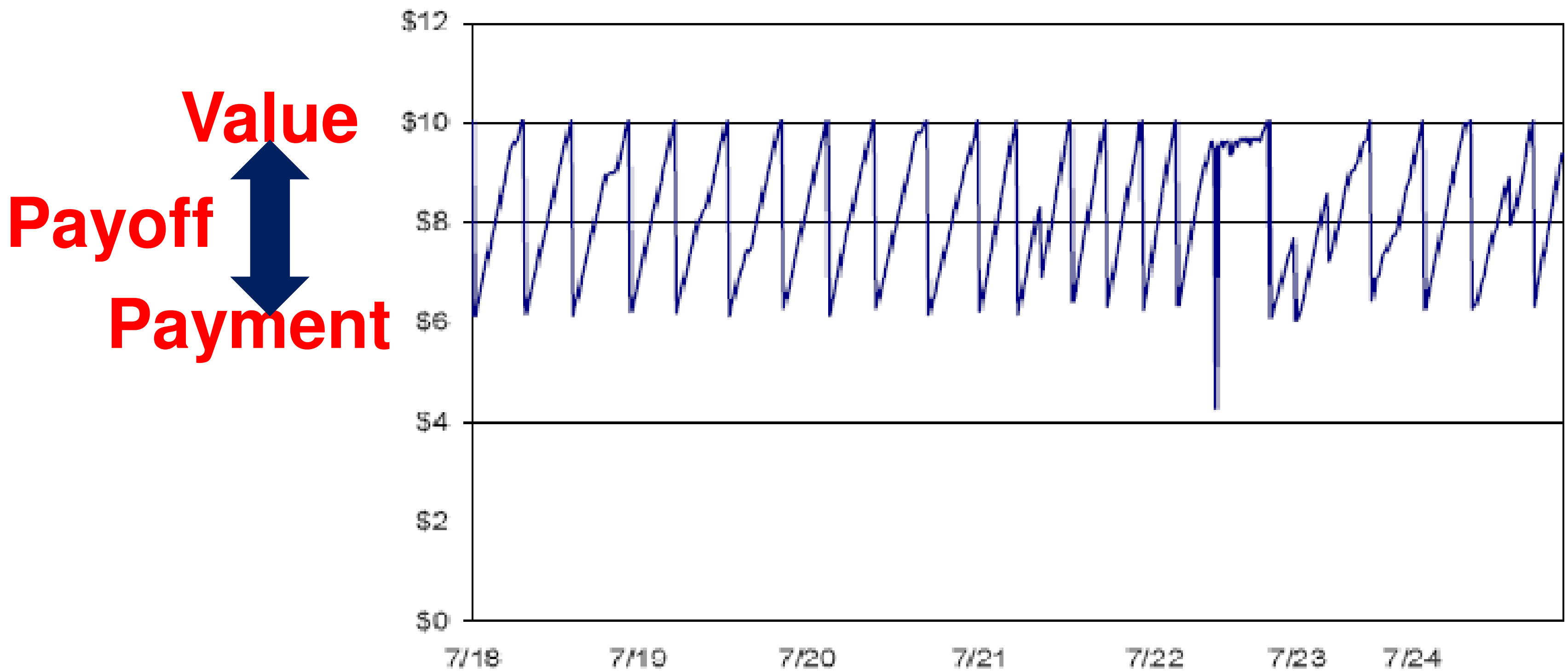
Thus, the payment of the last bidder who gets allocated a spot is the same as under GSP (and VCG):
0 if N ≥ K; $\#Click_N * b_{N+1}$ otherwise. (Note: K bidders; N Slots)

VCG is a generic truthful mechanism:

- **Allocation = the one that maximizes social welfare or total value (assuming value = bid)**
- **Price (i) = cost imposed by i on others**
  = total increase in others' value if i were to disappear.

Gaming the Overture System

- **Another notable aspect of Overture's auction design was that winning bids were posted**
- **Buyer's remorse versus truth-telling versus Nash's Equilibrium**
  - No equilibrium



(b) 1 week

Payoff=Value-Payment

Ranking by Expected Revenue

Low-Tech

- **Ranking by bid price only can also be gamed**
  - Get free branding experience; annoy consumers; ad spam
  - Goto/Overature addressed this need via editorial review

High-Tech

- **Google's introduced an auction mechanism which exploits the fact that advertisers bid (and pay) on a PPC basis rather than a CPM basis.**
  - Instead of allocating advertising slots in the decreasing order of bids, slots are allocated in the decreasing order of expected revenue.
  - This revenue is computed as the product of the advertiser's bid and the advertiser's expected click-through rate
    - an estimate of how likely the advertiser's ad is to be clicked on.

CPC Calculation

Payoff = Value – Price
Payoff = ValuePerClick – CPC

$$Bid_1 \times DQ_1 > Bid_2 \times DQ_2$$

For ad_1 to maintain it's current rank then Bid_1 needs to be at least:

$$Bid_1 \geq \frac{Bid_2 \times DQ_2}{DQ_1}$$

Ad Id	1. Receive Bid	2. Assess Quality	3. Calculate Rank	4. Set CPC Price
123	$5.80	10	$58.00	$1.71
ABC	$4.25	4	$17.00	$3.01
NOP	$2.00	6	$12.00	$0.51
TUV	$3.00	1	$3.00	$1.66
XYZ	$0.55	3	$1.65	Reserve Bid

Quality Score helps avoid Ad Spam

- Quality Score can prohibit advertisers from simply bidding high enough to show in the top position.
- E.g., Below, Cameron is bidding well above all of his competitors, he will show in the fourth position due to his low Quality Score.
- Determining Click Cost:
  - $\text{ChargeToAdvertiser}_i = (\text{AdQuality}_{i+1} / \text{AdQuality}_i)^* (\text{Bid}_{i+1}) + \$.01$
  - E.g., 1.6/10 + 0.1 = $0.17 Cost for the Mark (ad at ranked 1)

Rank by ECPM

Advertiser	Max CPC	Quality Score	AdRank	Position	Actual CPC
Mallory	$0.40	4	$0.4 x 4 = **1.6**	2	(1.2 / 4) + $.01 = $.31
Mark	$0.50	10	$0.50 x 10 = **5**	1	(1.6 / 10) + $.01 = $.17
Laura	$0.20	6	$0.20 x 6 = **1.2**	3	(1 / 6) + $.01 = $.17
Cameron	$2.00	0.5	$2.00 x .5 = **1**	4	(.8 / .5) + $.01 = $1.61
Alison	$0.05	16	$.05 x 16 = **.8**	5	(.2 / 2) + $.01 = $.11
Will	$0.10	2	$0.10 x 2 = **.2**	6	Minimum Bid

ECPM-based rankg and payment for CPC

- **Ranks ads based on Expected-Revenue$_{Ad}$ (aka ECPM)**
  - Google, MSN and, as of 2/2007, Yahoo use ECPM-based ranking

$$ECPM_{Ad} = CTR_{Ad} * Bid_{Ad}$$

$$ECPM_{Ad} = AdQualityIndex_{Ad} * Bid_{Ad}$$

PAY

$$CPC_{Ad@i} = \frac{AdQualityIndex_{Ad@i+1}}{AdQualityIndex_{Ad@i}} * Bid_{Ad@i+1}$$

Bid-to-Position Model | **ECPM-Ranking Model**

GSP is further complicated…

- **Additional factors change the properties of the auction mechanisms (making the whole process opaque)**
- **As a result, ad networks are providing some transparency, e.g., via keyword bidding tools**

$$Bid_{Ad}$$

$$Bid_{Ad} * CTR_{Ad}$$

$$Bid_{Ad} * CTR_{Ad} * ThrottleFactor$$

Electronic market mechanisms

- **The interplay of game theory and e-commerce is an exciting domain for future research.**
- **Progress in this area will require a combination of theoretical analysis, empirical studies, and simulation experiments.**
- **Better market designs will do a better job of matching buyers with sellers, ultimately enhancing the welfare of online advertising.**

Auction Mechanisms in Commercial Use

- **Rank ads by ECPM**
  - Price per click x clicks per impression = price per impression
- **Each bidder pays price determined by bidder below him**
  - Price = minimum price necessary to retain position
  - Motivated by engineering, not economics
- **Ranking using ECPM and Charge based on GSP**
  - Over 98% of Google's revenue comes from GSP-like auctions.
  - Over 50% of Yahoo!'s total revenue from GSP-like auctions.
  - MSN AdCenter (rank based on ECPM but charge based on GSP)

Summary: GSP is the workhorse of OA

$$ECPM_{Ad} = Quality_{Ad} * Bid_{Ad}$$

- **Revenue from GSP**
  - Over 98% of Google's revenue comes from GSP-like auctions.*
  - Over 50% of Yahoo!'s total revenue from GSP-like auctions.*
  - MSN AdCenter (rank based on ECPM but charge based on GSP)
- **The dominant strategy of GSP is not truth-telling**
  - But for Vickrey-Clark Groves (VCG) Auctions it is: each advertiser pays the externality (opportunity cost) he imposes on others
  - Publisher revenue: **GSP ≥ VCG**
  - NTL GSP is dominant in commercial settings
    - VCG is complicated to explain to typical advertisers; It is vulnerable to collusion by losing bidders; and shilling.
    - Static Equilibrium of GSP is locally envy-free; No advertiser can improve his payoff by exchanging bids with the advertiser in the slot above.

***[Edelman, B. et al, 2006]**

Auction Workshops/Conferences

- **Workshops, Conferences**
  - Annual ACM EC;
  - DIMACS Workshop Series
  - Trading Agent Competition (TAC)
  - WWW sessions and workshops
  - Game Theory

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